At least and *at most*: Scalar focus operators in context

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October 15, 2016

1. Introduction

A hallmark feature of the scalar operators *at least* and *at most* is their capacity to express ignorance on the part of the speaker: from an utterance of (1a) or (1b), a listener will typically conclude that the speaker is not certain exactly how many points LeBron scored (though this value is certainly no less/greater than 20 points).

(1) a. LeBron scored at least 20 points in last night's game.
b. LeBron scored at most 20 points in last night's game.

For as robust as these ignorance inferences are in simple unembedded contexts, they have a tendency to disappear in the presence of modals and nominal quantifiers: under their most salient (but not their only) interpretations, neither (2a) nor (2b) conveys any uncertainty regarding what is necessary or allowed. Rather, the most salient interpretations for these examples convey variation in what the speaker knows to be sufficient or permissible: from an utterance of (2a), a listener may conclude that 45 points will suffice, as will, say, 50 points, while an utterance of (2b) licenses the conclusions that (e.g.) submitting one singly authored and one co-authored abstract is permitted, as is submitting just one singly authored abstract.

(2) a. (In order to win the scoring title), LeBron <u>needs</u> to score at least 45 points in tonight's game.
b. One person <u>can</u> submit at most one abstract as sole author and one abstract as co-author (or two co-authored abstracts).

Such variation inferences are also favored when *at least* and *at most* occur in combination with quantificational DPs: from (3a), a listener may conclude that some player(s) scored 10 points, while others scored more than this, and from (3b), that it is acceptable (e.g.) to donate $2600 to one candidate and $400 apiece to several others. More generally, both (3a) and (3b) convey variation in the scalar values associated with the quantified individuals.

(3) a. <u>Every player</u> scored at least 10 points in last night's game.
b. Individuals can give to as many candidates as they want, so long as they give at most $2600 to <u>any single candidate</u> in an election cycle.

The question of exactly how *at least* and *at most* manage to express ignorance and variation has attracted considerable scrutiny. Early approaches to these data (Geurts & Nouwen 2007, Nouwen 2010) treated the above inferences as semantic entailments that directly figure into the truth-conditional meanings for the examples in (1)–(3). Following Büring (2008), much recent work has instead favored a pragmatic approach, in which the inferences are treated as conversational implicatures due to the interaction of these operators' core semantic properties with general pragmatic principles. There is also a sizeable and growing body of experimental literature on *at least* and *at most* (e.g., Cummins & Katsos 2010, McNabb & Penka 2014, Alexandropoulou 2015, Alexandropoulou et al. 2015, McNabb et al. 2016), which tends to support a pragmatic approach to such inferences while further probing their exact nature and distribution.

Although specific pragmatic accounts differ, sometimes widely, in their assumptions regarding the truth-conditional import of these operators as well as the mechanisms underlying implicature calculation, there is one point on which they are nearly unanimous, namely that the inferences licensed by *at least* and *at most* bear a striking, non-accidental resemblance to those arising with ordinary disjunction. That unembedded disjunctions convey speaker ignorance is well-known: from an utterance of (4a), a listener will conclude that the speaker is not certain exactly what Grover ate for dinner. When disjunction appears under a modal or a nominal quantifier, variation inferences likewise emerge: from an utterance of (4b), one may conclude that taking the final exam will be sufficient, as will writing a term paper, and from (4c), that some customer(s) ordered soup, while others ordered salad.

(4) a. Grover ate tuna, chicken, or duck for dinner.
b. (To pass this class), you <u>need to</u> take a final exam or write a term paper.
c. <u>Every customer</u> ordered a soup or a salad with dinner.

But capitalizing on this resemblance to construct the proper analogy to disjunction has proven to be a surprisingly tricky enterprise. In its simplest formulation, this analogy amounts to the view that *at least* and *at most* create *n*-ary, possibly infinite disjunctions over their associated scalar values and all higher or lower ones:

(5) a. LeBron scored at least 20 points.
≈ 'LeBron scored 20 points or 21 points or 22 points or … '
b. LeBron scored at most 20 points.
≈ 'LeBron scored 20 points or 19 points or 18 points or …'

While such a simple view correctly captures the truth-conditional behavior of *at least* several authors have observed that it appears to mischaracterize its associated ignorance inferences. Furthermore, it turns out that this simple view fails to account for the other main pragmatic effect observed for *at least*, namely its capacity to suspend certain upper-bounding inferences that would arise in its absence: unlike (1a), an utterance of (6) is reliably accompanied by the inference LeBron scored no more than 20 points.

(6) LeBron scored 20 points in last night's game.

This (perhaps) surprising failure of the simple view persists even under certain less simple views designed to better predict the ignorance inferences observed for *at least*. The problems presented by *at most* are more severe—without any further amendment, the simple view does not even adequately capture its truth-conditional effects. And for both *at least* and *at most*, it has been observed that the simple view is unable to account for the full range of interpretations that emerge in the presence of modals and nominal quantifiers.

My goal in this paper is to show that these problems, severe though they are, do not pose a unsurmountable obstacle for the analogy to disjunction in its simplest form. I will first argue that the simple view described above can indeed be maintained for *at least*, once more careful consideration is given to the scales that it operates over. Specifically, recognizing that these scales (i) may be fundamentally contextual in nature (vs. purely quantitative or otherwise conventional), and (ii) are in fact never ordered by entailment allows for a satisfactory account of *at least*'s pragmatic behavior. Although a correspondingly simple disjunctive view of *at most* cannot be maintained, I will demonstrate how its essential pragmatic insights may nonetheless be preserved within a truth-conditionally adequate treatment, one that takes its meaning to be negative, or exclusive, in character. Finally, I will explore how the resulting proposal may be fruitfully applied to some heretofore refractory interactions with modals and nominal quantifiers.

The structure of this paper is as follows: section 2 outlines the *n*-ary disjunction view and its obstacles in precise detail. Section 3 constitutes my attempted rehabilitation of this view. The results arrived at there are brought to bear on the variation readings observed for *at least* and *at most* in section 4. Section 5 concludes the paper.

2. *At least*, *at most*, and disjunction: A tantalizing but elusive analogy

2.1 At least *and* at most *as scalar focus operators*

Although most of the recent literature has concentrated on their occurrence as numeral modifiers, *at least* and *at most* are neither syntactically nor semantically restricted to combine with numerals. Rather, these expressions exhibit considerable flexibility in their syntactic positioning, as well as a parallel diversity in the range of ordered domains that they may operate over (Kay 1992, Krifka 1999, Geurts & Nouwen 2007). Some of this diversity is illustrated in (7).

(7) a. Grover ate at least [some]$_F$ of his dinner.
b. He was told that he had, at most, [several]$_F$ weeks left to live. (quantificational determiners)
c. When making a post that contains a banned word, at most you are allowed to reveal [the first letter and the length of the word]$_F$. (plural individuals)
d. Mabel won a [silver]$_F$ medal, at least. (rank order)
e. Back in my day, you had to at least [lose games]$_F$ before you got fired. (verb phrase meanings)

The examples in (7) also illustrate the constraining effects that focus placement has on the precise choice of scale and the final interpretation. Such effects are further seen in (8): whereas (8a) conveys that (for all the speaker knows), LeBron might have scored more than 20 points, (8b) instead conveys that some of LeBron's teammates might have scored 20 points. Similarly, whereas (8e) could be uttered to exclude Mabel's having waxed the car, (8f) might instead serve to exclude Mabel's having washed the truck.[1]

(8) a. At the very least, LeBron scored [20 points]$_F$ in last night's game.
b. At the very least, [LeBron]$_F$ scored 20 points in last night's game.
c. At the very least, LeBron scored 20 points in [last night's]$_F$ game.
d. Mabel at most [washed the car]$_F$.
e. Mabel at most [washed]$_F$ the car.
f. Mabel at most washed [the car]$_F$.

To simplify the following discussion, I assume that in the input to compositional interpretation, *at least* and *at most* always attach to proposition-denoting constituents, some portion of which bears F-marking. Thus, I take the relevant structural representation of (1a) and (1b), repeated below as (9a), to be (9b), though I see no obstacle to defining suitably type-shifted meanings that will allow these operators to compose directly with their syntactic scopes.

(9) a. LeBron scored at least/at most 20 points in last night's game.
b. at least/at most [LeBron scored [20 points]$_F$ in last night's game]

I also make the standard assumption that the semantic import of F-marking, $[\![\cdot]\!]_f$, is to evoke a set of alternative semantic values, or focus alternatives (Rooth 1985, Fox & Katzir 2011). Like other scalar focus operators (e.g., *only* and *even*), *at least* and *at most* presuppose that these alternatives form partially ordered scales. Some of the scales that figure into the preceding examples are illustrated below.

[1] In sentence-initial position, the "pure" scalar use of *at least* that is the topic of this paper is expressed with *at the* (*very) least*. Sentence-initial *at least* instead expressed the evaluative scalar use discussed by Kay (1992):

(i) At least Mabel won a [silver]$_F$ medal.

In (i), the presence of *at least* conveys a "settling for less" inference—Mabel's winning gold would have been preferable to her winning silver.

(10) a. ⟦ LeBron scored [20 points]$_F$ in last night's game ⟧$_f$
= { … ⟦ LeBron scored 21 points in last night's game ⟧ ≻
⟦ LeBron scored 20 points in last night's game ⟧ ≻
⟦ LeBron scored 19 points in last night's game ⟧ ≻ … }

b. ⟦ Grover ate [some]$_F$ of his dinner ⟧$_f$ = { ⟦ G. ate all of his dinner ⟧ ≻
⟦ G. ate most of his dinner ⟧ ≻
⟦ G. ate some of his dinner ⟧ }

c. ⟦ Mabel won a [silver]$_F$ medal ⟧$_f$ = { ⟦ Mabel won a gold medal ⟧ ≻
⟦ Mabel won a silver medal ⟧ ≻
⟦ Mabel won a bronze medal ⟧ }

2.2 at least *as* n-*ary disjunction: Truth-conditional vacuity and weakening*

The truth-conditional behavior of *at least* is, I think, relatively uncontroversial. In particular, whether *at least* has any impact at all on the truth-conditional meaning depends upon the nature of its associated scale. With (apparently) quantitative scales ordered by semantic entailment, *at least* appears to be truth-conditionally vacuous: (11a) and (11b) will both be true if Grover ate just some of his dinner, but also if he ate most or even all of it.

(11) a. Grover ate some of his dinner.
b. Grover ate at least [some]$_F$ of his dinner.

With non-entailment scales, such as the rank order scale in (12), *at least* has a weakening effect: only (12b) is additionally compatible with Mabel's having won gold.

(12) a. Mabel won a silver medal.
b. Mabel won a [silver]$_F$ medal, at least.

Krifka (1999) insightfully observes that these differing effects can be unified by the assumption that *at least* applies to the proposition in its scope, or its prejacent, and returns the *n*-ary disjunction over the prejacent and all focus alternatives that are ordered more highly than it. When these alternatives are ordered by entailment, the resulting disjunction will in fact be semantically equivalent to the weakest disjunct, i.e., to the prejacent itself: the disjunction SOME ∨ MOST ∨ ALL in (13) is equivalent to the prejacent SOME.

(13) ⟦ at least [Grover ate [some]$_F$ of his dinner] ⟧
= ⟦ Grover ate some of his dinner ⟧ ∨ SOME ∨
⟦ Grover ate most of his dinner ⟧ ∨ MOST ∨
⟦ Grover ate all of his dinner ⟧ ALL
≡ ⟦ Grover ate some of his dinner ⟧ ≡ SOME

When the alternatives are not ordered by entailment, *at least* may have an observable effect on truth-conditional meaning: the disjunction SILVER ∨ GOLD in (14) is weaker than the prejacent SILVER, which alone entails ¬GOLD.

(14) $[\![$ at least [Mabel won a $[\text{silver}]_F$ medal] $]\!]$
= $[\![$ Mabel won a silver medal $]\!]$ ∨   SILVER ∨
$[\![$ Mabel won a gold medal $]\!]$ GOLD

Whether or not *at least* has a truth-conditional effect with numerals, which at various points have been argued both to entail and not to entail their lower scalemates, ultimately depends on whether a one-sided, entailment semantics (e.g., Horn 1972) or a two-sided, non-entailment semantics (e.g., Horn 1992, Geurts 2006, Kennedy 2013) is adopted for the bare numerals themselves:

(15) a. $[\![$ at least [LeBron scored $[20 \text{ points}]_F$] $]\!]$ (one-sided/entailment)
$= \text{PTS} \geq 20 \vee \text{PTS} \geq 21 \vee \text{PTS} \geq 22 \vee \ldots$
$\equiv \exists n[\, n \in \{ m \mid m \geq 20 \} \,\&\, \text{PTS} \geq n \,]$
$\equiv \text{PTS} \geq 20$

b. $[\![$ at least [LeBron scored $[20 \text{ points}]_F$] $]\!]$ (two-sided/non-entailment)
$= \text{PTS} = 20 \vee \text{PTS} = 21 \vee \text{PTS} = 22 \vee \ldots$
$\equiv \exists n[\, n \in \{ m \mid m \geq 20 \} \,\&\, \text{PTS} = n \,]$
$\equiv \text{PTS} \geq 20$

A very simple treatment of *at least* as an existential quantifier over propositions, with a quantificational domain consisting of the prejacent and all higher focus alternatives, is provided in (16).

(16) $[\![\text{at least S}]\!] = \lambda w.\, \exists q[\, q \in \{ p \mid p \in [\![\text{S}]\!]_f \,\&\, p \geq [\![\text{S}]\!] \} \,\&\, q(w) \,]$

Krifka briefly notes that *at least* may signal a speaker's ignorance, but does not explicitly account for this observation with his proposal. As previously mentioned, much recent work has explored the possibility that these inferences are derived pragmatically as conversational implicatures (e.g., Büring 2008, Cummins & Katsos 2010, Mayr 2013, Schwarz 2013, 2016a, Kennedy 2015, Nouwen 2015 for accounts in terms of the maxim of Quantity, Coppock & Brockhagen 2013 for an account (roughly) in terms of the maxim of Quality, and Cohen & Krifka 2014 for a quite different account in terms of meta-speech acts). Although there is diversity even amongst these accounts, I will try to abstract away from it here, and in the next subsection present a very general picture in terms of the neo-Gricean approach to implicature calculation.

2.3 at least *as* n-*ary disjunction: Too much ignorance and implicature unsuspension*

Under the neo-Gricean approach (Grice 1967/1975, 1978, Horn 1972), listeners draw additional inferences about the speaker's communicative intent in uttering some sentence *S* by reasoning about the members of a formally defined set of pragmatic competitors to *S*, which we may abbreviate as COMP(*S*). Regarding the neo-Gricean derivation of Quantity implicatures, there is a fairly established view of how this reasoning proceeds (Sauerland 2004, Fox 2007), enough so to have been dubbed the Standard Recipe by Geurts (2011). The Standard Recipe comes in three steps: in the first, a listener infers from a speaker's utterance of *S* that she is not certain of the truth of any stronger competitor *S′* belonging to COMP(*S*). Such an inference, of the form ¬BEL(*S′*), amounts to a weak Quantity implicature regarding *S′*. In the next step, the listener may additionally take the speaker to be competent with respect to this stronger competitor *S′*—she is either certain of its truth or certain of its falsity, BEL(*S′*) ∨ BEL(¬*S′*). Step three follows directly from the results of steps one and two: the listener may come to infer that the speaker is certain that *S′* is false. This final inference, of the form BEL(¬*S′*), amounts to a strong Quantity implicature regarding *S′* (note that it entails the corresponding weak implicature).

A compelling feature of the Standard Recipe is the way that it derives both pragmatic upper-bounding and ignorance inferences as subspecies of Quantity implicatures. Consider the ordinary disjunction in (17). Following Sauerland (2004), we may assume that its competitor set contains, apart from (17) itself, three stronger competitors, namely the two individual disjuncts, along with their conjunction:

(17) Grover had tuna or chicken for dinner.

COMP(17) = { T ∨ C , T , C , T & C }[2]

The Standard Recipe derives straightaway the speaker's certainty that the conjunction is false, which amounts to the canonical upper-bounding, or exclusivity, implicature observed for disjunction:

(i)	BEL(T ∨ C)	(Quality)
(ii)	¬BEL(T & C), ¬BEL(T), ¬BEL(C)	(Weak Quantity)
(iii)	BEL(T & C) ∨ BEL(¬(T & C))	(Competence)
(iv)	BEL(¬(T & C))	(Strong Quantity)

Regarding the disjuncts, the Standard Recipe yields weak Quantity implicatures, but here, strengthening either implicature under the speaker's presumed competence results in a contradiction. Specifically, a strong Quantity implicature regarding the one disjunct T will, in conjunction with the Quality implicature in (i), entail the speaker's certainty that the other disjunct C is true, BEL(C). But this entailment contradicts the weak Quantity implicature regarding C in (ii):

[2] Here and in what follows, the asserted meaning appears underlined within the competitor set.

(v) $\textsc{Bel}(\textsc{t}) \vee \textsc{Bel}(\neg\textsc{t})$ (Competence)
(vi) $\textsc{Bel}(\neg\textsc{t})$ (Strong Quantity: $\bot$)

Instead, we must reject the presumption of speaker competence regarding T, which is tantamount to presuming the speaker's ignorance regarding T. Parallel reasoning yields an ignorance implicature regarding the other disjunct C:

(vii) $\neg(\textsc{Bel}(\textsc{t}) \vee \textsc{Bel}(\neg\textsc{t}))$ (Ignorance)
$\equiv \textsc{Poss}(\textsc{t})\ \&\ \textsc{Poss}(\neg\textsc{t})$
(viii) $\textsc{Poss}(\textsc{c})\ \&\ \textsc{Poss}(\neg\textsc{c})$ (Ignorance)

In (17), the individual disjuncts T and C form a pair of **symmetric stronger competitors**—strengthening the first's weak Quantity implicature under competence will contradict the second's weak Quantity implicature, and vice versa. A general feature of the Standard Recipe is that it derives ignorance implicatures for any pair of symmetric competitors (Fox 2007, Schwarz 2013, 2016a).

Extending the above picture from binary to *n*-ary disjunction requires that the Standard Recipe derive an ignorance implicature for each of the *n* disjuncts. This can be achieved with a competitor set that contains the *n* individual disjuncts and is closed under disjunction:[3]

(18) Grover ate tuna, chicken or duck for dinner.

COMP(18) = { $\underline{\textsc{t} \vee \textsc{c} \vee \textsc{d}}$,
$\textsc{t} \vee \textsc{c}$, D, symmetric
$\textsc{t} \vee \textsc{d}$, C, symmetric
$\textsc{c} \vee \textsc{d}$, T } symmetric

Since *or* can equivalently be described as an existential quantifier over propositions, we may alternatively say that the competitors to an *n*-ary disjunction consist of all those existential statements based on stronger, or narrower (possibly even singleton), quantificational domains (Alonso-Ovalle 2006, Chierchia 2013). The competitor set for ordinary disjunction is thereby shaped by the question of why the speaker did not choose a stronger quantificational domain for her existential statement:

$$\text{COMP}(18) = \{\ \lambda w.\, \exists p[\, p \in Q\ \&\ p(w)\,]\ \mid\ Q \subseteq \{\textsc{t}, \textsc{c}, \textsc{d}\}\ \}$$

As demonstrated below for the symmetric competitors T and $\textsc{c} \vee \textsc{d}$, the Standard Recipe indeed yields an ignorance implicature regarding the individual disjuncts. Just as before, a strong Quantity implicature regarding T will, in conjunction with Quality, entail the speaker's certainty that its symmetric

[3] I am omitting any possible competitors to (18) formed through conjunction over the individual disjuncts.

counterpart is true, BEL(C ∨ D). The resulting contradiction necessitates the rejection of speaker competence regarding T:

(i)	$\text{BEL}(\text{T} \vee \text{C} \vee \text{D})$	(Quality)
(ii)	$\neg\text{BEL}(\text{T}), \neg\text{BEL}(\text{C} \vee \text{D})$	(Weak Quantity)
(iii)	$\text{BEL}(\text{T}) \vee \text{BEL}(\neg\text{T})$	(Competence)
(iv)	$\text{BEL}(\neg\text{T})$	(Strong Quantity: $\bot$)
(v)	$\neg(\text{BEL}(\text{T}) \vee \text{BEL}(\neg\text{T}))$ $\equiv \text{POSS}(\text{T}) \;\&\; \text{POSS}(\neg\text{T})$	(Ignorance)
(vi)	$\text{POSS}(\text{C}) \;\&\; \text{POSS}(\neg\text{C})$	(Ignorance)
(vii)	$\text{POSS}(\text{D}) \;\&\; \text{POSS}(\neg\text{D})$	(Ignorance)

Parallel reasoning yields ignorance implicatures regarding the other disjuncts C and D.

At this point, a very simple account of *at least*'s capacity to express speaker ignorance suggests itself. Having already claimed that *at least* forms an *n*-ary disjunction over its prejacent and all higher focus alternatives, it seems entirely reasonable to impute the same rich set of competitors to an *at least*-sentence as was just assumed for ordinary disjunction. The competitor set will thus contain the prejacent and all higher focus alternatives, and will furthermore be closed under disjunction, to again generate the full set of existential statements based on stronger quantificational domains. This is illustrated below for (1a) *LeBron scored at least 20 points* (*in last night's game*), given a two-sided semantics for bare numerals.

(19) at least [LeBron scored $[20 \text{ points}]_F$] (two-sided/non-entailment)

$$
\begin{aligned}
\text{COMP}(19) = \{\; & \underline{\exists n[\, n \in \{\, m \mid m \geq 20 \,\} \;\&\; \text{PTS} = n \,]}\,, \\
& \equiv \text{PTS} \geq 20 \\
& \left.\begin{aligned}
& \exists n[\, n \in \{\, m \mid m \geq 20 \,\} - \{\, 20 \,\} \;\&\; \text{PTS} = n \,]\,, \\
& \equiv \text{PTS} \geq 21 \\
& \exists n[\, n \in \{\, 20 \,\} \;\&\; \text{PTS} = n \,]\,, \\
& \equiv \text{PTS} = 20
\end{aligned}\right] \\
& \left.\begin{aligned}
& \exists n[\, n \in \{\, m \mid m \geq 20 \,\} - \{\, 21 \,\} \;\&\; \text{PTS} = n \,]\,, \\
& \equiv \text{PTS} = 20 \vee \text{PTS} \geq 22 \\
& \exists n[\, n \in \{\, 21 \,\} \;\&\; \text{PTS} = n \,]\,, \\
& \equiv \text{PTS} = 21
\end{aligned}\right] \\
& \ldots \;\} \\
= \{\; & \exists n[\, n \in Q \;\&\; \text{PTS} = n \,] \;\big|\; Q \subseteq \{\, m \mid m \geq 20 \,\} \;\}
\end{aligned}
$$

The Standard Recipe will derive an ignorance implicature regarding each of *at least*'s "disjuncts" in the expected fashion. Together, they yield the inference that for every n greater than or equal to 20, the speaker considers it possible that LeBron scored exactly n points:

$\text{POSS}(\text{PTS} = 20)\ \&\ \text{POSS}(\neg\, \text{PTS} = 20)$
$\text{POSS}(\text{PTS} = 21)\ \&\ \text{POSS}(\neg\, \text{PTS} = 21)$
…
————

$\forall n[\ n \geq 20 \rightarrow (\text{POSS}(\text{PTS} = n)\ \&\ \text{POSS}(\neg\, \text{PTS} = n))\ ]$
(Ignorance re: prejacent and higher focus alternatives)

Unfortunately, this simple account suffers from two problems (at least!), which I describe below. While the first is fairly well-recognized, the second has gone virtually unnoticed in the recent literature on *at least*.

The first problem is that this simple-minded extension would seem to produce too many ignorance implicatures—unlike ordinary n-ary disjunction, *at least* does not generally express total ignorance regarding each of its "disjuncts", and nor, for that matter, does *at most* (Schwarz 2013, 2016a, Ander-Mendia 2015, see also Alexandropoulou et al. 2015, Nouwen 2015 for parallel observations regarding variation inferences). Certainly, there need be no inference that the speaker considers highly implausible or impossible alternatives to be possible: as a rule, an utterance of *LeBron scored at least 20 points in last night's game* will not implicate that he might have scored, say, 650 points (no basketball player has ever scored more than 100 points in a single NBA game), nor will an utterance of *The New England Patriots scored at most 3 points in last night's game* implicate that the team might have scored just 1 point (which is not a possible final score in American football). More generally, although both *at least* and *at most* do reliably bring about ignorance implicatures regarding their prejacents, they need not produce such an implicature about any individual higher or lower alternative. This privileged status of the prejacent relative to the other alternatives is shown by the continuations in (20) and (21): one cannot subsequently express certainty that the prejacent is false, but one can subsequently express certainty that a particular higher or lower alternative is.

(20) Grover ate at least some of his dinner, though I'm sure that…
a. #he didn't eat just some of it.
b. he didn't eat all of it.
c. #he didn't eat more than (just) some of it.

(21) At most, he is a colonel, though I'm sure that…
a. #he is not a colonel.
b. he is at least a lieutenant. (entails that he is not a sargeant)
c. #he is nothing lower than a colonel.

In contrast, no individual disjunct in an ordinary disjunction is privileged above the rest: one cannot subsequent express certainty that <u>any</u> of the disjuncts is false.

(22) Grover ate tuna, chicken, or duck for dinner, though I'm sure that…
a. #he didn't eat tuna.
b. #he didn't eat chicken.
c. #he didn't eat duck.

A possible response to this problem that has been explored is to somehow restrict the set of pragmatic competitors evoked by *at least*. Since the Standard Recipe depends upon the presence of symmetric competitors to produce ignorance implicatures, we might account for the difference between (20)/(21) and (22) by claiming that the competitor set evoked by *at least* is more impoverished, and exhibits less symmetry, than the set evoked by ordinary *n*-ary disjunction. In particular, we might assume that amongst the competitors evoked by *at least*, it is only the prejacent that finds a symmetric counterpart:

(23) at least [LeBron scored [20 points]$_F$]

COMP(23) = { <u>PTS ≥ 20</u> ,
PTS ≥ 21 , PTS = 20 , symmetric
(…) }

The above three-membered competitor set is discussed by Büring (2008) and explicitly adopted by Kennedy (2015). Schwarz (2013, 2016a) and Nouwen (2015) endorse a much larger competitor set, which has substantially more members than just these three; crucially, none of the additional competitors that these authors contemplate participates in a symmetric pairing. In either case, the Standard Recipe will still yield an ignorance implicature about the prejacent, but ignorance regarding each individual higher focus alternative is no longer predicted. Rather, a single ignorance implicature regarding the disjunction over all higher alternatives will be produced:

POSS(PTS = 20) & POSS(¬ PTS = 20) (Ignorance re: prejacent)
POSS(PTS ≥ 21) & POSS(¬ PTS ≥ 21) (Ignorance re: disjunction over higher focus alternatives)

Although this type of solution is able to account for the contrasts seen in (20) and (21), it suffers from a variety of conceptual and empirical worries. Importantly, all of the above proposals rely, at least in part, upon Horn's (1972) substitution strategy to arrive at their postulated competitor sets. Under this strategy, pragmatic competitors are generated by systematic replacement of the scalar items in a sentence with members of their lexically-specified substitution classes, or Horn sets. Kennedy (2015) posits that the Horn set for *at least 20* includes the comparative *more than 20* and the bare numeral *20* (under a two-sided semantics), while Schwarz (2013, 2016a) assumes two different Horn sets, one consisting of the modifiers *at least* and *exactly/only*, and another consisting of the bare numerals.[4] (Both of these authors are chiefly concerned with occurrences of *at least* as a numeral modifier, as is Nouwen (2015).)

(24) LeBron scored $\left\{\begin{array}{c}\text{at least 20}\\ \text{more than 20}\\ \text{20}\end{array}\right\}$ points in last night's game.

(25) LeBron scored $\left\{\begin{array}{c}\text{at least}\\ \text{exactly / only}\end{array}\right\}\left\{\begin{array}{c}\text{…19}\\ \text{20}\\ \text{21…}\end{array}\right\}$ points in last night's game.

In doing so, these proposals end up generating the necessary competitor sets via an entirely different strategy from reasoning about stronger quantificational domains—note that the resulting competitors under either proposal invariably constitute existential statements over stronger domains, and so are also generated by the latter strategy. Quantificational-domain reasoning has already been called upon to account for the pragmatic behavior of ordinary disjunction, as already seen, as well as so-called epistemic indefinites such as German *irgendein* (Kratzer & Shimoyama 2002) and Spanish *algún* (Alonso-Ovalle & Menendez-Benito 2010). The observations in (20)–(22) show that the parallelism between *at least* and *at most* and ordinary disjunction is not absolute. Nevertheless, if it is possible to generate in a unitary fashion the stronger-domain competitors that these items have been claimed to consistently evoke, then it seems to me preferable to do so. Another worry about the substitution method concerns its compatiblity with the syntactic flexibility and scalar diversity exhibited by *at least*: as illustrated by (26) and (27), not every occurrence of *at* (*the very*) *least* can be grammatically replaced with *more than* or *exactly/only*.

(26) $\left\{\begin{array}{c}\text{At the very least,}\\ \text{*More than,}\\ \text{*Exactly / *Only,}\end{array}\right\}$ LeBron scored [20 points]$_F$ in last night's game.

(27) Back in my day, you had to $\left\{\begin{array}{c}\text{at least}\\ \text{*more than}\\ \text{*exactly / }^{??}\text{only}\end{array}\right\}$ [lose games]$_F$ before you got fired.

A final, somewhat different worry is specific to Schwarz's (and Nouwen's) proposed competitor set, which includes many stronger competitors that lack any symmetric counterparts. As Schwarz himself observes, the presence of such competitors allows for the derivation of unattested strong Quantity implicatures, since the results are not guaranteed to contradict any of the weak Quantity

[4] Nouwen's (2015) proposal constitutes an interesting hybrid, in that some of the competitors arise via reasoning about stronger quantificational domains, while other are generated by Horn-style lexical substitution. As he points out, the final set of competitors given by the two strategies is in fact equivalent to Schwarz's.

implicatures regarding other competitors (see Schwarz's work for a possible solution to this problem).[5]

The second problem is in some sense the opposite of the first—it turns out that when *at least* operates over a quantitative scale ordered by semantic entailment, the Standard Recipe fails to produce any ignorance implicatures at all! Instead, it incorrectly derives upper-bounding implicatures. This is a rather curious result, especially given that *at least* is one of Horn's (1972) implicature suspension devices—by virtue of conveying ignorance regarding higher alternatives, it simultaneously calls off the upper-bounded interpretations that otherwise arise in its absence. Whereas an utterance of (28) will in most discourse situations implicate that Grover ate <u>only</u> some of his dinner, (29a) lacks this inference altogether, as do (29b) and (29c), which illustrate some of the other suspenders identified by Horn.

(28) Grover ate some of his dinner.

(29) a. Grover ate at least some of his dinner.
b. Grover ate some of his dinner, if not all of it.
c. Grover ate some of his dinner, perhaps even all of it.

Within the neo-Gricean approach, the upper-bounded interpretation of (28) is accounted for with a competitor set that includes the stronger scalar alternatives built with *most* and *all*. The Standard Recipe then derives the upper-bounding inferences of (28) as strong Quantity implicatures:

COMP(28) = { <u>SOME</u> , MOST , ALL }

(i)	BEL(SOME)	(Quality)
(ii)	¬BEL(MOST) , ¬BEL(ALL)	(Weak Quantity)
(iii)	BEL(MOST) ∨ BEL(¬MOST), BEL(ALL) ∨ BEL(¬ALL)	(Competence)
(iv)	BEL(¬MOST), BEL(¬ALL)	(Strong Quantity)

Consider now the competitor set evoked by (29a). By hypothesis, this competitor set includes the prejacent SOME, alongside the stronger alternatives MOST and ALL, and is furthermore closed under disjunction. Because of the entailment relationships that exist amongst these three propositions, the

[5] Still another proposal for deriving the ignorance implicatures of *at least* and *at most* comes from Coppock & Brockhagen (2013). Working within the framework of inquisitive semantics, these authors suggest that *at least* and *at most* introduce questions, or issues, into their discourse contexts. They further assume a version of the maxim of Quality (their maxim of Interactive Sincerity) which obliges a speaker to raise an issue only if that issue is unresolved in her epistemic state. Schwarz (2016b) observes that although this proposal yields speaker uncertainty, it fails to predict true ignorance regarding any of the alternatives (i.e., that the speaker considers that alternative to be possible), not even regarding the prejacent. Nouwen (2015) makes essentially this same observation, albeit in a different theoretical context.

hypothesized competitor set for (29a) is in fact equivalent to the set { SOME , MOST , ALL }, i.e., to the competitor set for (28):

$$\text{COMP(29a)} = \{ \underset{\equiv \text{ SOME}}{\underline{\text{SOME} \vee \text{MOST} \vee \text{ALL}}}, \underset{\equiv \text{ MOST}}{\text{MOST} \vee \text{ALL}}, \underset{\equiv \text{ SOME}}{\text{SOME} \vee \text{ALL}}, \underset{\equiv \text{ SOME}}{\text{SOME} \vee \text{MOST}}, \text{SOME}, \text{MOST}, \text{ALL} \}$$

$$= \{ \underline{\text{SOME}}, \text{MOST}, \text{ALL} \}$$

This equivalence guarantees that the Standard Recipe cannot distinguish between (28) and (29a)—rather than deriving ignorance implicatures for (29a), the Standard Recipe instead re-introduces, or "unsuspends", the strong upper-bounding implicatures BEL(¬MOST) and BEL(¬ALL). Somewhat perversely, the very same logical property that underlies the truth-conditional vacuity of *at least* in (29a), viz., the equivalence of A ∨ B and A whenever B entails A, impedes an account of the ignorance inferences associated with this example.[6]

Whether or not the problem of implicature unsuspension afflicts numerals again depends on whether a two-sided (cf. (19)) or a one-sided semantics is adopted for for bare numerals: (30) shows that if the one-sided semantics is adopted, the unsuspension problem arises.

(30) at least [LeBron scored [20 points]$_F$] (one-sided/entailment)

$$\text{COMP(30)} = \{ \underset{\equiv\ \text{PTS} \geq 20}{\underline{\exists n[\, n \in \{ m \mid m \geq 20 \} \;\&\; \text{PTS} \geq n \,]}},$$

$$\underset{\equiv\ \text{PTS} \geq 21}{\exists n[\, n \in \{ m \mid m \geq 20 \} - \{ 20 \} \;\&\; \text{PTS} \geq n \,]},$$

$$\underset{\equiv\ \text{PTS} \geq 20}{\exists n[\, n \in \{ 20 \} \;\&\; \text{PTS} \geq n \,]},$$

$$\underset{\equiv\ \text{PTS} \geq 20}{\exists n[\, n \in \{ m \mid m \geq 20 \} - \{ 21 \} \;\&\; \text{PTS} \geq n \,]},$$

$$\underset{\equiv\ \text{PTS} \geq 21}{\exists n[\, n \in \{ 21 \} \;\&\; \text{PTS} \geq n \,]},$$

$$\underset{\equiv\ \text{PTS} \geq 20}{\exists n[\, n \in \{ m \mid m \geq 20 \} - \{ 22 \} \;\&\; \text{PTS} \geq n \,]},$$

[6] Note that the unsuspension problem does not arise for non-entailment scales, e.g., for rank orders—the upper-bounding inference conveyed by *Mabel won a silver medal* constitutes a semantic entailment, not a conversational implicature. While this inference is suppressed for *Mabel won at least a silver medal*, due to the truth-conditional weakening effected by *at least*, application of the Standard Recipe does not reintroduce any upper-bounding, but rather correctly yields ignorance implicatures.

$\exists n[\ n \in \{ 22 \}\ \&\ \textsc{pts} \geq n\ ]$,
$\equiv\ \textsc{pts} \geq 22$
… }

$= \{\ \underline{\textsc{pts} \geq 20}\ ,\ \textsc{pts} \geq 21\ ,\ \textsc{pts} \geq 22\ ,\ \ldots\ \}$
$= \{\ \exists n[\ n \in Q\ \&\ \textsc{pts} \geq n\ ]\ \big|\ Q \subseteq \{\ m \mid m \geq 20\ \}\ \}$

BEL(¬PTS ≥ 21), BEL(¬PTS ≥ 22), … (Strong Quantity)

A possible explanation for why this problem has not been sufficiently appreciated in the literature on *at least* lies in the disproportionate attention that has been paid to this operator's numeral uses—as the difference between (19) and (30) shows, adopting a two-sided numeral semantics skirts the problem. Indeed, several authors have either explicitly (e.g., Kennedy 2015) or implicitly (e.g., Nouwen 2015) endorsed a two-sided semantics for bare numerals. But as shown for (29a), the unsuspension problem generalizes to any scale that is ordered by semantic entailment. It is worth noting that the unsuspension problem is not crucially linked to reasoning about stronger quantificational domains—for instance, it will also arise within the most charitable extensions of Kennedy's and Nouwen's scalar substitution approaches to non-numeral scales.

2.3 at most *as* n*-ary disjunction: Upper-bounding and existential inferences*

Turning to *at most*, Krifka (1999) already observes that a disjunctive treatment parallel to *at least* fails to capture its truth-conditional behavior. Such a parallel treatment is illustrated in (31), where *at most* applies to its prejacent to return the *n*-ary disjunction consisting of the prejacent and all lower focus alternatives.

(31) $[\![$ at most S $]\!] = \lambda w.\ \exists q[\ q \in \{\ p \mid p \in [\![\ S\ ]\!]_f\ \&\ \underline{p \leq [\![\ S\ ]\!]}\ \}\ \&\ q(w)\ ]$

One immediate problem arises when *at most* operates over quantitative entailment scales. For such examples, the meaning in (31) will not yield any upper-bounding inference—since none of the alternatives entails an upper bound, neither will their disjunction:

(32) a. $[\![$ at most [LeBron scored [20 points]$_F$] $]\!]$ (one-sided/entailment)
$= \textsc{pts} \geq 20 \vee \textsc{pts} \geq 19 \vee \ldots\ \textsc{pts} \geq 2 \vee \textsc{pts} \geq 1$
$\equiv \textsc{pts} \geq 1$ (compatible with LeBron's scoring 40 points)
b. $[\![$ at most [he has [several]$_F$ weeks left to live] $]\!]$
= SEVERAL ∨ SOME
≡ SOME (compatible with his having many weeks left to live)

The upper-bounding inference that *at most* encodes is clearly semantic in nature, rather than pragmatic. Note, for instance, that the inference cannot be suspended (Blok 2015, Alexandropoulou et al. 2015 on Greek *to poli* 'at most'):

(33) a. LeBron scored 20 points in last night's game, perhaps more.
b. #LeBron scored at most [20 points]$_F$ in last night's game, perhaps more.

(34) a. He was told that he had several weeks left to live, if not more.
b. #He was told that he had, at most, [several]$_F$ weeks left to live, if not more.

A common response to this problem is to add the upper-bounding entailment directly into the meaning of *at most*, via some additional maximization operator (Coppock & Brockhagen 2013, Kennedy 2015), or for some scales (e.g., numerals), into the meanings of the focus alternatives themselves (Nouwen 2015): as shown in (35), the meaning in (31) in combination with a two-sided numeral semantics yields upper-bounded truth conditions.

(35) ⟦ at most [LeBron scored [20 points]$_F$] ⟧ (two-sided/non-entailment)
$= \text{PTS} = 20 \vee \text{PTS} = 19 \vee \ldots \text{PTS} = 2 \vee \text{PTS} = 1$
$\equiv 1 \leq \text{PTS} \leq 20$

However, neither of these solutions on its own accounts for the second problem that a disjunctive treatment of *at most* faces, namely that as an existential quantifier, *at most* should yield an existential entailment that some focus alternative is true. As Krifka notes, *at most* does often give rise to <u>some kind</u> of existential inference, as evidenced by its ability to introduce discourse referents: *At most three boys left*. *They found the play boring*. But unlike its upper-bounding cousin, this lower-bounding inference is merely pragmatic, and is easily suspended (Blok 2015, Schwarz et al. 2012, McNabb 2015; see also Sanford & Moxey 2004):

(36) a. #LeBron scored 20 points in last night's game, and it's even possible that he didn't score any points at all.
b. LeBron scored at most [20 points]$_F$ in last night's game, and it's even possible that he didn't score any points at all.

(37) a. #Mabel won a silver medal, if indeed she won anything at all.
b. At most, Mabel won a [silver]$_F$ medal, if indeed she won anything at all.

A possible response to this problem is to stipulate that the scales that *at most* operates over include a "zero", or "null", alternative (e.g., Coppock & Brockhagen 2013): as seen in (38), the resulting truth conditions for (1b) *LeBron scored at most 20 points* (*in last night's game*) will no longer entail that he scored some (positive) number of points.

(38) ⟦ at most [LeBron scored [20 points]$_F$] ⟧ (two-sided / non-entailment)
= PTS = 20 ∨ PTS = 19 ∨ … PTS = 2 ∨ PTS = 1 ∨ PTS = 0
≡ 0 ≤ PTS ≤ 20 (compatible with LeBron's not scoring any points)

Such a response may be plausible for numerals, but strikes me as much less so for many other scales, such as the ones evoked in (39). We do not ordinarily conceive of rank orders as in (39a) as encompassing a null value, and constructing such an alternative for (39b) would seem to require the (to my mind) dubious notion of a null individual. And for (39c), making room for the possibility that Mabel did literally nothing is surely incorrect. Rather, what must be accommodated is the possibility that she didn't do anything relevant, e.g., anything that was asked of her—she may very well have eaten lunch, taken a nap, gone to a movie, etc.

(39) a. At most, Mabel won a [silver]$_F$ medal.
b. Mabel at most washed [the car]$_F$.
c. Mabel at most [washed the car]$_F$.

Furthemore, treating the possibility that LeBron scored no points in (38) as "just another disjunct" does not account for the non-entailed existential inference that Krifka and others have detected with *at most*. If anything, an ignorance implicature regarding the zero alternative would instead be expected. But the contrast between (40a) and (40b) indicates that the inference regarding whether LeBron scored any points at all patterns differently from the inference regarding whether he scored 20 points.

(40) a. LeBron scored at most [20 points]$_F$ in last night's game, and it's even possible that he didn't score any points at all.
b. #LeBron scored at most [20 points]$_F$ in last night's game, and it's even possible that he scored (exactly) 20 points.

Rather, the existential inference patterns with other strong upper-bounding implicatures in its capacity to undergo suspension: the difference seen in (40) mirrors exactly the one illustrated in (41) between the upper-bounding and ignorance implicatures of ordinary disjunction.

(41) a. Grover had tuna or chicken for dinner, possibly even both.
b. #Grover had tuna or chicken for dinner, possibly even tuna.

Krifka's own solution to these problems is to reject entirely the disjunctive treatment of *at most*. Rather than forming a disjunction over its prejacent and all lower alternatives, *at most* instead serves to negate, or exclude, all those focus alternatives ordered more highly than the prejacent. A very simple rendition of this proposal is provided in (42), and some of the resulting truth conditions in (43) and (44)—under either a one-sided or a two-sided semantics for bare numerals, the truth-conditional meaning of (1b) merely requires that LeBron not have scored 21 points or 22 points or 23 points or … .[7]

(42) $[\![\text{at most S}]\!] = \lambda w.\ \underline{\neg}\exists q[\ q \in \{\ p \mid p \in [\![\text{S}]\!]_f\ \&\ \underline{p > [\![\text{S}]\!]}\ \}\ \&\ q(w)\ ]$

(43) $[\![\text{at most [he has [several]}_F \text{ weeks left to live]}]\!]$
$= \neg(\textsc{quite-a-few} \vee \textsc{many} \vee \textsc{a-great-many} \vee \ldots)$

(44) a. $[\![\text{at most [LeBron scored [20 points]}_F \text{]}]\!]$ (one-sided/entailment)
$= \neg(\textsc{pts} \geq 21 \vee \textsc{pts} \geq 22 \vee \textsc{pts} \geq 23 \vee \ \ldots)$
$\equiv \neg\exists n[\ n \in \{\ m \mid m \geq 21\ \}\ \&\ \textsc{pts} \geq n\ ]$
$\equiv \neg\textsc{pts} \geq 21$

b. $[\![\text{at most [LeBron scored [20 points]}_F \text{]}]\!]$ (two-sided/non-entailment)
$= \neg(\textsc{pts} = 21 \vee \textsc{pts} = 22 \vee \textsc{pts} = 23 \vee \ldots)$
$\equiv \neg\exists n[\ n \in \{\ m \mid m \geq 21\ \}\ \&\ \textsc{pts} = n\ ]$
$\equiv \neg\textsc{pts} \geq 21$

Interestingly, this treatment of *at most* has not been widely pursued in the subsequent literature—so far as I know, the only recent account that countenances a purely exclusive meaning for *at most* is Penka's (2015) proposed decomposition of *at most* into *at least* and an abstract antonymizing operator. Perhaps one reason for this reluctance is that this alternative treatment leaves unclear how to account for the ignorance implicatures accompanying *at most*, given that its meaning is no longer disjunctive. Nor does it shed any further light on how its pragmatic existential inference should be derived.

To summarize, the simple view of *at least* as forming the *n*-ary disjunction over its prejacent and all higher focus alternatives provides a welcome characterization of its truth-conditional import, but not of its pragmatic behavior—in some cases, it predicts too many ignorance implicatures, while for entailment scales, it fails to predict any at all, and instead yields unattested upper-bounding implicatures. The corresponding view of *at most* fails even to correctly characterize its truth-conditional behavior. An alternative view, which treats *at most* as essentially negative, or exclusive, in character, is able to do so, but in severing the link to ordinary disjunction, it does not immediately suggest an account of *at most*'s pragmatic effects.

3. Getting (just) enough ignorance

In this section, I present my own solutions to the problems identified previously. In section 3.1, I return to the proper characterization of the ignorance

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[7] The meaning in (42) is simplified for expository purposes, and consequently yields the correct results only for totally ordered scales. The following meaning is suitable for both totally and partially ordered scales:

(i) $[\![\text{at most S}]\!] = \lambda w.\ \forall q[\ q \in \{\ p \mid p \in [\![\text{S}]\!]_f\ \&\ p(w)\ \} \rightarrow q \leq [\![\text{S}]\!]\ ]$

inferences accompanying *at least* (and *at most*). I argue that the *n*-ary disjunction treatment is indeed compatible with the only-partial ignorance that this operator expresses. Specifically, the privileged status of the prejacent relative to its other scalar alternatives may be seen as reflecting the role of the discourse context in enumerating the scale. In section 3.2, I take up the problem of implicature unsuspension with *at least*. I review an analogous problem that has been identified for ordinary disjunction, and propose that its solution be adopted here as well. Section 3.3 addresses *at most*—I show there how reasoning about stronger quantificational domains, which drives the Standard Recipe's account of ordinary disjunction, may be combined with a purely ecxlusive treatment of this operator to yield ignorance implicatures.

3.1 Scales and contextual relevance

Recall that the first problem for the simple *n*-ary disjunction view of *at least* was its inability to distinguish the prejacent's ignorance implicature from those regarding any individual higher or lower alternative—unlike ordinary disjunction, *at least* and *at most* need not express total ignorance. I would like to suggest that a solution to this problem can be found in the scalar diversity that was previously observed for these operators. The previous examples illustrated various purely quantitative scales, such as the nominal quantifiers, the numerals, and the structured domains of individuals and verb-phrase meanings, as well as non-quantitative but still conventionalized scales, such as rank orders. In fact, the scales that *at least* and *at most* operate over may also be fundamentally non-conventional in nature, defined only relative to a particular discourse context (Kay 1992, see also Fillmore et al. 1988 on *let alone*, Matsumoto 1997 on *if not*).

Consider the examples in (45): if I utter (45a) upon entering the casino, then I may very well succeed in conveying that the lucky player might have thrown 7 or 11—although the relevant scale for this example is built upon the numerals, a non-canonical ordering is induced by the rules of the game. In (45b), a patently non-conventional scale is presupposed, in which evidence for one's sexual orientation is somehow ranked more highly than evidence for one's thirst.

(45) a. (Simplified rules for the dice game craps: if a player throws a 2, 3, or 12 on her first roll, she loses her bet. If a player throws a 7 or 11 on her first roll, she wins her bet. If a player throws a 4, 5, 6, 8, 9, or 10 on her first roll, she gets subsequent rolls/chances to win her bet.)
(Upon seeing a player collect her winning bet)
She at least threw $[4, 5, 6, 8, 9, \text{or } 10]_F$ on her first roll.

b. But hanging out in a gay bar is not evidence that one is gay.
At most, it is evidence of $[\text{thirstiness and a desire to get drunk}]_F$.
(Geurts & Nouwen 2007)

The examples in (46) demonstrate that both the scalar values and the ordering amongst them may be quite *ad hoc*: whereas (46a) may implicate the possibility that Pete has already arrived at Amsterdam, (46b) will instead implicate the possibility that Pete has not yet departed Amsterdam.

(46) a. (Uttered about Pete's outbound trip from Boston to Utrecht, with stops in Reykjavik and Amsterdam)
At the very least, he's made it to $[\text{Reykjavik}]_F$ by now.
b. (Uttered about the return trip, with the same stops in reverse order)
At most, he's made it to $[\text{Reykjavik}]_F$ by now.

The examples in (47) rely upon conventional scales, but in each case, the preceding linguistic context imposes quite severe restrictions on their memberships. B's response in (47a) may implicate the possibility that Pete likes Nora's mother, but certainly will not implicate the possibility that he likes, say, Barack Obama, whereas (47b) may entail that Mabel didn't mow the lawn, but in all likelihood will not entail that she didn't take a nap.

(47) a. A: Does Pete like any of Nora's relatives?
B: He at least likes $[\text{her father}]_F$.
b. I doubt that Mabel finished her chores—at most, she $[\text{washed the car}]_F$.

Of course, this sort of contextual dependence is a well-known property of other scalar focus operators, such as *even* and *only*:

(48) A: Does Fred eat sushi?
B: He sure does! He even eats $[\text{squid}]_F$!

(49) John brought Tom, Bill, and Harry to the party, but he only introduced $[\text{Bill}]_F$ to Sue. (Rooth 1996)

Rooth (1992, 1996) incorporate such dependence into his earlier theory of focus interpretation by proposing that the semantics of focus does not directly furnish these operators with their scales. Rather, focus operators are essentially anaphoric to a contextually provided scale. Focus placement acts to constrain, or perhaps to indicate, the admissible resolutions of this anaphoric dependency. In (50), I show the results of incorporated Rooth's proposal into our previous meaning for *at least*.

(50) $[\![\text{at least}_C\ S]\!] = \lambda w.\ \exists q[\ q \in \{\ p \mid p \in \underline{C}\ \&\ p \geq [\![S]\!]\ \}\ \&\ q(w)\ ]$
Presuppositions: (i) $C \subseteq [\![S]\!]_f$, (ii) $[\![S]\!] \in C$, and (iii) $|C| > 1$

The subscripted variable *C* in (50) indicates that *at least* is now anaphoric to a contextually provided scale, and it is this scale that *at least* existentially quantified over. The role of focus is to trigger certain presuppositions about the value of *C*: (i) *C* forms a subset of the prejacent's focus alternatives, (ii) *C* includes the prejacent, and (iii) apart from the prejacent, *C* contains at least one other focus alternative. Presupposition (i) guarantees that the identity of the contextually given scale is at least partially recoverable from the utterance itself. Presupposition (ii) guarantees that the prejacent is a member of its scale—by choosing to evoke a certain set of focus alternatives with the prejacent, the speaker explicitly signals the prejacent's relevance in context. Presupposition

(iii) guarantees that the scale is non-trivial—at least one other focus alternative is contextually relevant.

My suggestion is that Rooth's proposal provides all that is needed to solve the "too much ignorance" problem. That is, I maintain the claim that the pragmatic competitors to an *at least*-sentence consist of its full set of stronger-domain competitors, relativized to the contextually provided scale *C*. The competitor set therefore contains the prejacent and all contextually relevant higher focus alternatives, and is furthermore closed under disjunction. To illustrate, consider (1a) again, and suppose that *C* consists of the focus alternatives $\text{PTS} = n$ for $10 \leq n < 100$.

(51) at least_C [LeBron scored [20 points]$_F$] (two-sided/non-entailment)
$C = \{ \text{PTS} = n \mid 10 \leq n < 100 \}$

Relative to this value for *C*, (51) will be true so long as LeBron scored (exactly) *n* points, for $20 \leq n < 100$. The stronger pragmatic competitors are now all those existential statements based on subsets of this contextually relevant range of values:

$$
\begin{aligned}
\text{COMP}(51) = \{\ & \underline{\exists n[\, n \in \{ m \mid 20 \leq m < 100 \} \,\&\, \text{PTS} = n \,]}\,, \\
& \equiv\ 20 \leq \text{PTS} < 100 \\
& \exists n[\, n \in \{ m \mid 20 \leq m < 100 \} - \{ 20 \} \,\&\, \text{PTS} = n \,]\,, \\
& \equiv\ 21 \leq \text{PTS} < 100 \\
& \exists n[\, n \in \{ 20 \} \,\&\, \text{PTS} = n]\,, \\
& \equiv\ \text{PTS} = 20 \\
& \exists n[\, n \in \{ m \mid 20 \leq m < 100\} - \{ 21 \} \,\&\, \text{PTS} = n \,]\,, \\
& \equiv\ \text{PTS} = 20 \vee 22 \leq \text{PTS} < 100 \\
& \exists n[\, n \in \{ 21 \} \,\&\, \text{PTS} = n]\,, \\
& \equiv\ \text{PTS} = 21 \\
& \ldots\ \} \\
= \{\ & \exists n[\, n \in Q \,\&\, \text{PTS} = n \,] \mid Q \subseteq \{ m \mid 20 \leq m < 100 \}\ \}
\end{aligned}
$$

Applied to this competitor set, the Standard Recipe only yields ignorance implicatures regarding the prejacent and each contextually relevant higher alternative, as opposed to total ignorance:

$\text{POSS}(\text{PTS} = 20)\ \&\ \text{POSS}(\neg\text{PTS} = 20)$
$\text{POSS}(\text{PTS} = 21)\ \&\ \text{POSS}(\neg\text{PTS} = 21)$
…
$\text{POSS}(\text{PTS} = 99)\ \&\ \text{POSS}(\neg\text{PTS} = 99)$

$\forall n[\ 20 \leq n < 100 \rightarrow (\text{POSS}(\text{PTS} = n)\ \&\ \text{POSS}(\neg\text{PTS} = n))\ ]$
(Ignorance re: prejacent and contextually relevant higher alternatives)

The lack of any ignorance implicatures regarding alternatives that are mutually regarded as implausible or impossible (e.g., PTS = 650) follows immediately, under the plausible assumption that such alternatives are necessarily irrelevant.

Of course, the identity of *C* will in general not be so uniquely determined. In the most extreme (but not so uncommon) case of contextual indeterminacy, a listener will know nothing about the speaker's intended value for *C* apart from what is guaranteed by the focus-induced presuppositions. Relative to such a context, the most that a listener may infer is what the Standard Recipe will yield under any admissible value for *C*. Below, I list a handful of other admissible values for *C* in (51), along with the ignorance implicatures that they precipitate:

$C = \{\ \text{PTS} = 19\ ,\ \text{PTS} = 20\ ,\ \text{PTS} = 21\ \}$

$\text{POSS}(\text{PTS} = 20)\ \&\ \text{POSS}(\neg\text{PTS} = 20)$
$\text{POSS}(\text{PTS} = 21)\ \&\ \text{POSS}(\neg\text{PTS} = 21)$ (entails $\text{POSS}(\text{PTS} \geq 21)$)

$C = \{\ \text{PTS} = 18\ ,\ \text{PTS} = 20\ ,\ \text{PTS} = 22\ \}$

$\text{POSS}(\text{PTS} = 20)\ \&\ \text{POSS}(\neg\text{PTS} = 20)$
$\text{POSS}(\text{PTS} = 22)\ \&\ \text{POSS}(\neg\text{PTS} = 22)$ (entails $\text{POSS}(\text{PTS} \geq 21)$)

$C = \{\ \text{PTS} = 10\ ,\ \text{PTS} = 20\ ,\ \text{PTS} = 30\ ,\ \text{PTS} = 40\ ,\ \text{PTS} = 50\ \}$

$\text{POSS}(\text{PTS} = 20)\ \&\ \text{POSS}(\neg\text{PTS} = 20)$
$\text{POSS}(\text{PTS} = 30)\ \&\ \text{POSS}(\neg\text{PTS} = 30)$ (entails $\text{POSS}(\text{PTS} \geq 21)$)
$\text{POSS}(\text{PTS} = 40)\ \&\ \text{POSS}(\neg\text{PTS} = 40)$ (entails $\text{POSS}(\text{PTS} \geq 21)$)
$\text{POSS}(\text{PTS} = 50)\ \&\ \text{POSS}(\neg\text{PTS} = 50)$ (entails $\text{POSS}(\text{PTS} \geq 21)$)

Under any admissible value for *C*, the Standard Recipe will yield an ignorance implicature regarding the prejacent PTS = 20, whose presence in *C* is guaranteed by presupposition (ii). No individual higher focus alternative is guaranteed to be present in *C*, and so for each such alternative, there is an admissible value for *C* under which the Standard Recipe will not yield an ignorance implicature. However, if the presence of *at least* is non-trivial, then there must be some higher focus alternative(s) in *C*. So no matter the value assigned to *C*, the Standard Recipe should yield an ignorance implicature regarding some higher focus alternative. In other words, it follows under every admissible value for *C* that there is some higher focus alternative that the speaker considers to be possible, or equivalently, that the speaker considers it possible that some higher focus alternative is true:

$$\text{POSS}(\text{PTS} = 20)\ \&\ \text{POSS}(\neg\,\text{PTS} = 20)$$
$$\text{POSS}(\text{PTS} \geq 21)\ \&\ \text{POSS}(\neg\,\text{PTS} \geq 21)$$

In this way, the ignorance implicature regarding the prejacent comes to have a distinguished status relative to those regarding any individual higher focus alternatives.

3.2 At least, *embedded exhaustification, and implicature re-suspension*

Our second problem for the *n*-ary disjunction view concerned the Standard Recipe's failure to produce ignorance implicatures whenever *at least* operates over a quantitative entailment scale. For such examples, the Standard Recipe instead ends up "unsuspending" certain upper-bounding implicatures. Chierchia et al. (2009, 2011) have observed that this same unsuspension problem arises for ordinary disjunction, which is another one of Horn's implicature suspension devices. For instance, all of the disjunctions in (52) serve to suspend an upper-bounding implicature—indeed, the overall interpretation of (52a) closely resembles that of (29a) *Grover ate at least some of his dinner*.

(52) a. Grover ate some or (even) most of his dinner.
b. Grover ate [[tuna or chicken] or both] for dinner.
c. LeBron scored 20 or 21 points in last night's game.

For the very same reasons previously identified for (29a), the Standard Recipe will not yield ignorance implicatures regarding the individual disjuncts in (e.g.) (52a). This is again because the pragmatic competitors for (52a) are ultimately aligned by semantic strength:

$$\text{COMP(52a)} = \{\ \underset{\equiv\ \text{SOME}}{\underline{\text{SOME} \vee \text{MOST}}}\ ,\ \text{SOME}\ ,\ \text{MOST}\ ,\ \underset{\equiv\ \text{MOST}}{\text{SOME}\ \&\ \text{MOST}}\ \}$$
$$= \{\ \underline{\text{SOME}}\ ,\ \text{MOST}\ \}$$

$\text{BEL}(\neg\,\text{MOST})$ (Strong Quantity)

The examples in (52) are problematic for another reason, which is that they appear to run afoul of a constraint on the felicitous use of disjunction first identified by Hurford (1974). Hurford's constraint bans disjunctions in which one disjunct entails the other—the infelicitous examples in (53) show the constraint in action.

(53) a. #Mary saw an animal or a dog.
b. #That painting is of a man or a bachelor.
c. #The value of x is different from 6 or greater than 6.

The problem is that the examples in (52) are perfectly acceptable, despite their apparent violation of Hurford's constraint.

Chierchia et al. show that both of the problems posed by (52) may be solved at once with the assumption that in each example, the first disjunct is parsed with a covert scalar focus operator, *exh*(*aust*). Under their proposal, the relevant structural representation of (52a) is (54).

(54) [*exh* [Grover ate [some]$_F$ of his dinner]] or [Grover ate most of his dinner]

The *exh* operator applies to its prejacent to return the conjunction of the prejacent with the negations of its stronger focus alternatives. This results in an upper-bounded, or exhaustive, truth-conditional meaning for the prejacent relative to its focus alternatives (see Fox 2007 for a more refined definition):

(55) $[\![\, exh\ \text{S}\,]\!] \;=\; \lambda w.\ [\![\, \text{S}\,]\!](w)\ \&\ \forall q[\ q \in \{\, p \mid p \in [\![\, \text{S}\,]\!]_f\ \&\ p(w)\,\} \rightarrow [\![\, \text{S}\,]\!] \subseteq q\ ]$

(56) $[\![\, exh$ [Grover ate [some]$_F$ of his dinner] $]\!]$
= SOME & ¬MOST & ¬ALL
≡ SOME & ¬MOST

The first disjunct in (54) now entails ¬MOST, and so the two disjuncts are logically incompatible. The violation of Hurford's constraint in (52) is therefore illusory, since the presence of *exh* serves to disrupt any entailment relationships between the disjuncts. The truth-conditional meaning of (54) is (SOME & ¬MOST) ∨ MOST, which is still equivalent to SOME. But a welcome side effect is that the pragmatic competitors to (54) are no longer aligned by semantic strength—rather, the two disjuncts SOME & ¬MOST and MOST now form a pair of symmetric competitors:[8]

COMP(54) = { <u>(SOME & ¬MOST) ∨ MOST</u> ,
≡ SOME
SOME & ¬MOST , MOST , symmetric
(SOME & ¬MOST) & MOST }
≡ ⊥
= { <u>SOME</u> , SOME & ¬MOST , MOST }

[8] Suppose that the weak Quantity implicature ¬BEL(MOST) is strengthened under competence to BEL(¬MOST). In conjunction with the Quality implicature BEL(SOME), this strong Quantity implicature will entail BEL(SOME & ¬MOST), which contradicts the other weak Quantity implicature ¬BEL(SOME & ¬MOST). Analogous reasoning leads to the rejection of competence regarding the disjunct SOME & ¬MOST.

POSS(SOME & ¬MOST) & POSS(¬(SOME & ¬MOST)) (Ignorance)
POSS(MOST) & POSS(¬MOST) (Ignorance)

In this way, the presence of *exh* in the first disjunct manages to re-suspend the unattested upper-bounding implicatures in (52) in favor of the desired ignorance implicatures.[9]

It should come as no surprise that the unsuspension problem identified earlier for *at least* arises for ordinary disjunction, given their analogous behavior. It is perhaps also unsurprising that the same solution can be provided to both manifestations of the problem. This amounts to the claim that *at least* never operates over truly quantitative scales ordered by semantic entailment. (Nor, as we shall see shortly, does *at most*.) I will assume that in all such putative instances, the *exh* operator occurs somewhere in the prejacent, where it associates with the same F-marked constituent as does *at least*. The relevant structural representation for (29a) is shown in (57) (see Krifka 1991, Wold 1996 on the compositional interpretation of such structures; see also Crnic 2012, Kilbourn-Ceron 2016 for other cases of embedded *exh*).

(57) at least$_C$ [*exh* [Grover ate [some]$_F$ of his dinner]]

Exhaustification of *at least*'s prejacent in (57) disrupts any entailment relationships amongst its focus alternatives, and so effectively transforms an entailment scale into a non-entailment scale. At the same time, this exhaustification does not affect the overall truth-conditional meaning of (57)—the *n*-ary disjunction over the exhaustified prejacent and all higher exhaustified alternatives is still equivalent to SOME:

(58) ⟦ *exh* [Grover ate [some]$_F$ of his dinner] ⟧$_f$
= { ⟦ *exh* [Grover ate [all]$_F$ of his dinner] ⟧ ≻ ALL ≻
⟦ *exh* [Grover ate [most]$_F$ of his dinner] ⟧ ≻ MOST & ¬ALL ≻
⟦ *exh* [Grover ate [some]$_F$ of his dinner] ⟧ } SOME & ¬MOST

(59) ⟦ at least$_C$ [$_S$ *exh* [Grover ate [some]$_F$ of his dinner]] ⟧ (C = ⟦ S ⟧$_f$)[10]
= ⟦ *exh* [Grover ate [some]$_F$ of his dinner] ⟧ ∨ (SOME & ¬MOST) ∨
⟦ *exh* [Grover ate [most]$_F$ of his dinner] ⟧ ∨ (MOST & ¬ALL) ∨
⟦ *exh* [Grover ate [all]$_F$ of his dinner] ⟧ ALL
≡ ⟦ Grover ate some of his dinner ⟧ ≡ SOME

---

[9] Chierchia et al. introduce the *exh* operator in the course of developing their grammatical theory of upper-bounding implicatures, a direct competitor to the neo-Gricean approach. I continue to employ the latter for the purposes of implicature calculation.

[10] For the remainder of this paper, I will assume that the contextually provided scale to *at least* is identical to the focus semantic value of its prejacent.

Just as it did for ordinary disjunction, the presence of *exh* in *at least*'s prejacent solves the unsuspension problem. As shown below, the competitor set for (57) is organized into pairs of symmetric stronger competitors, for which the Standard Recipe yields ignorance implicatures.

COMP(57) = { (SOME & ¬MOST) ∨ (MOST & ¬ALL) ∨ ALL ,
≡ SOME

(MOST & ¬ALL) ∨ ALL , SOME & ¬MOST , symmetric
≡ MOST

(SOME & ¬MOST) ∨ ALL , MOST & ¬ALL , symmetric

(SOME & ¬MOST) ∨ (MOST & ¬ALL) , ALL } symmetric
≡ SOME & ¬ALL

POSS(SOME & ¬MOST) & POSS(¬(SOME & ¬MOST)) (Ignorance)
POSS(MOST & ¬ALL) & POSS(¬(MOST & ¬ALL)) (Ignorance)
POSS(ALL) & POSS(¬ALL) (Ignorance)

Regarding numerals, I follow Spector (2013) in assuming that their one-sided semantics is basic. Evidence for a two-sided semantics reflects their "strong preference for being in the scope of *exh*", perhaps due to their intrinsically activating focus alternatives. The relevant structural representation for (1a) therefore includes an occurrence of *exh*:

(60) $[\![$ at least$_C$ [$_S$ *exh* [LeBron scored [20 points]$_F$]] $]\!]$ ($C = [\![$ S $]\!]_f$)
= (PTS ≥ 20 & ¬PTS ≥ 21 & …) ∨ (PTS ≥ 21 & ¬PTS ≥ 22 & …) ∨
≡ PTS = 20 ≡ PTS = 21
(PTS ≥ 22 & ¬PTS ≥ 23 & …) ∨ …
≡ PTS = 22
≡ $\exists n[\, n \in \{ m \mid m \geq 20 \}$ & PTS $= n\,]$
≡ PTS ≥ 20

As shown in (60), the *exh* operator again has the effect of converting a one-sided entailment scale into a two-sided non-entailment scale.

3.3 at most*, negated disjunction, and stronger quantificational domains*

Turning to *at most*, recall that a purely exclusive treatment, while truth-conditionally adequate, does not immediately yield an account of its pragmatic effects. Our earlier exclusive meaning for *at most* is repeated in (61), altered slightly to reflect its anaphoric dependence on the discourse context, while the structural representation for (1b) in (62) now assumes the presence of *exh* in the prejacent.

(61) $[\![\text{at most}_C \text{ S}]\!] = \lambda w.\ \neg\exists q[\ q \in \{\ p \mid p \in C\ \&\ p > [\![\text{S}]\!]\ \}\ \&\ q(w)\ ]$
symmetric
Presuppositions: (i) $C \subseteq [\![\text{S}]\!]_f$, (ii) $[\![\text{S}]\!] \in C$, and (iii) $|C| > 1$

(62) $[\![\text{at most}_C\ [_S\ exh\ [\text{LeBron scored [20 points]}_F]]\ ]\!]$ $\quad (C = [\![\text{S}]\!]_f)$
$= \neg(\text{PTS} = 21 \lor \text{PTS} = 22 \lor \text{PTS} = 23 \lor \ldots)$
$\equiv \neg\exists n[\ n \in \{\ m \mid m \geq 21\ \}\ \&\ \text{PTS} = n\ ]$
$\equiv \neg\text{PTS} \geq 21$

One problem concerns the ignorance that *at most* conveys. In fact, my proposed solution to this problem is relatively straightforward—since *at most* creates a negated existential statement, the competitor set that it evokes may still be shaped by reasoning about stronger quantificational domains. The crucial difference, of course, is that under negation, strengthening is achieved by widening, rather than narrowing, the domain of existential quantification. As shown in (62), *at most* excludes all (contextually relevant) focus alternatives ordered more highly than the prejacent by negating the *n*-ary disjunction over those alternatives. Its stronger competitors should then exclude not only all of these alternatives, but some lower one(s) as well:

COMP(62) = { $\underline{\neg\exists n[\ n \in \{\ m \mid m \geq 21\ \}\ \&\ \text{PTS} = n\ ]}$,
$\equiv \neg\text{PTS} \geq 21$

$\neg\exists n[\ n \in \{\ m \mid m \geq 21\ \} \cup \{\ 20\ \}\ \&\ \text{PTS} = n\ ]$,
$\equiv \neg(\text{PTS} \geq 21 \lor \text{PTS} = 20)$

$\neg\exists n[\ n \in \mathbb{N} - \{\ 20\ \}\ \&\ \text{PTS} = n\ ]$,
$\equiv \neg(\text{PTS} \geq 21 \lor 1 \leq \text{PTS} \leq 19)$

$\neg\exists n[\ n \in \{\ m \mid m \geq 21\ \} \cup \{\ 19\ \}\ \&\ \text{PTS} = n\ ]$,
$\equiv \neg(\text{PTS} \geq 21 \lor \text{PTS} = 19)$

$\neg\exists n[\ n \in \mathbb{N} - \{\ 19\ \}\ \&\ \text{PTS} = n\ ]$,
$\equiv \neg(\text{PTS} \geq 20 \lor 1 \leq \text{PTS} \leq 18)$

…

$\neg\exists n[\ n \in \mathbb{N}\ \&\ \text{PTS} = n\ ]$
$\equiv \neg\text{PTS} \geq 1$ }

$= \{\ \neg\exists n[\ n \in Q\ \&\ \text{PTS} = n\ ] \mid Q \supseteq \{\ m \mid m \geq 21\ \}\ \}$

One of the stronger competitors to (62) excludes all alternatives ordered more highly than the prejacent, along with the prejacent itself. Another such competitor excludes all higher alternatives, as well as all lower alternatives. Finally, the strongest competitor to (62) will be the one that excludes all of the (contextually relevant) focus alternatives.

How does the Standard Recipe fare with respect to this competitor set? Consider the first two stronger competitors listed above. The Standard Recipe derives the weak Quantity implicatures in (ii) regarding both of these competitors, which, in conjunction with the Quality implicature in (i), yield the entailments in (iii). For instance, if the speaker is certain that LeBron didn't score more than 20 points (Quality), but she is not certain that he didn't score 20 or more points (weak Quantity), then she must not be certain that he didn't score (exactly) 20 points, which is equivalent to considering 20 points to be a possibility:

(i)	$\text{B}\textsc{el}(\neg\textsc{pts} \geq 21)$	(Quality)
(ii)	$\neg\text{B}\textsc{el}(\neg(\textsc{pts} \geq 21 \vee \textsc{pts} = 20))$, $\neg\text{B}\textsc{el}(\neg(\textsc{pts} \geq 21 \vee 1 \leq \textsc{pts} \leq 19))$	(Weak Quantity)
(iii)	$\neg\text{B}\textsc{el}(\neg\textsc{pts} = 20)$, $\neg\text{B}\textsc{el}(\neg 1 \leq \textsc{pts} \leq 19)$ $\equiv \text{P}\textsc{oss}(\textsc{pts} = 20)$ $\equiv \text{P}\textsc{oss}(1 \leq \textsc{pts} \leq 19)$	(Entailments of (i) and (ii))

Next, suppose that the speaker is competent with respect to one of these competitors, e.g., $(\neg(\textsc{pts} \geq 21 \vee \textsc{pts} = 20)$. Given the Quality implicature in (i), this presumption of competence can be restated as in (iv), and it will derive the strong Quantity implicature in (v). But the resulting implicature will entail $\text{B}\textsc{el}(\neg 1 \leq \textsc{pts} \leq 19)$, which contradicts one of the entailments in (iii)! I.e., if the speaker is certain that LeBron scored (exactly) 20 points (strong Quantity), then she must also be certain that he didn't score anywhere from 1 to 19 points, which contradicts our earlier conclusion that she considers this a possibility:

(iv)	$\text{B}\textsc{el}(\neg\textsc{pts} = 20) \vee \text{B}\textsc{el}(\textsc{pts} = 20)$	(Competence)
(v)	$\text{B}\textsc{el}(\textsc{pts} = 20)$	(Strong Quantity: $\perp$)

So the presumption of speaker competence must be rejected, and we instead arrive at an ignorance implicature regarding the prejacent:

(vi)	$\neg(\text{B}\textsc{el}(\neg\textsc{pts} = 20) \vee \text{B}\textsc{el}(\textsc{pts} = 20))$ $\equiv \text{P}\textsc{oss}(\textsc{pts} = 20)\ \&\ \text{P}\textsc{oss}(\neg\textsc{pts} = 20)$	(Ignorance)

More generally, the two stronger competitors form a symmetric pair, and analogous reasoning about the other symmetric competitors eventually yields ignorance implicatures regarding every (contextually relevant) lower alternative:

(vii) $\text{P}\textsc{oss}(\textsc{pts} = 19)\ \&\ \text{P}\textsc{oss}(\neg\textsc{pts} = 19)$
(viii) $\text{P}\textsc{oss}(\textsc{pts} = 18)\ \&\ \text{P}\textsc{oss}(\neg\textsc{pts} = 18)$
…
―――――
$\forall n[\ n \leq 20 \rightarrow (\text{P}\textsc{oss}(\textsc{pts} = n)\ \&\ \text{P}\textsc{oss}(\neg\,\textsc{pts} = n))\ ]$
(Ignorance re: prejacent and (contextually-relevant) lower alternatives)

As with *at least*, in extreme cases of contextual indeterminacy, the most that a listener may infer is what the Standard Recipe will yield under <u>any</u> admissible value for C, namely an ignorance implicature regarding the prejacent, and one regarding the disjunction of all lower focus alternatives:

$$\text{POSS}(\text{PTS} = 20)\ \&\ \text{POSS}(\neg\text{PTS} = 20)$$
$$\text{POSS}(\text{PTS} \leq 19)\ \&\ \text{POSS}(\neg\text{PTS} \leq 19)$$

It bears mention that the occurrence of *exh* in (62) is a necessary component of the above proposal. Its omission results in the mirror image of the unsuspension problem encountered for *at least*—without *exh*, the pragmatic competitors to (1b) will instead be aligned by semantic strength, resulting this time in an unattested lower-bounding implicature:

(63) $[\![$ at most$_C$ [LeBron scored [20 points]$_F$] $]\!]$ $\quad (C = [\![\text{S}]\!]_f)$

$$= \neg(\text{PTS} \geq 21 \vee \text{PTS} \geq 22 \vee \text{PTS} \geq 23 \vee \ldots)$$
$$\equiv \neg\exists n[\ n \in \{ m \mid m \geq 21 \}\ \&\ \text{PTS} \geq n\ ]$$
$$\equiv \neg\text{PTS} \geq 21$$

$$\text{COMP}(63) = \{\ \underline{\neg\exists n[\ n \in \{ m \mid m \geq 21 \}\ \&\ \text{PTS} \geq n\ ]}\ ,$$
$$\equiv \neg\text{PTS} \geq 21$$
$$\neg\exists n[\ n \in \{ m \mid m \geq 21 \} \cup \{ 20 \}\ \&\ \text{PTS} \geq n\ ]\ ,$$
$$\equiv \neg\text{PTS} \geq 20$$
$$\neg\exists n[\ n \in \mathbb{N} - \{ 20 \}\ \&\ \text{PTS} \geq n\ ]\ ,$$
$$\equiv \neg\text{PTS} \geq 1$$
$$\neg\exists n[\ n \in \{ m \mid m \geq 21 \} \cup \{ 19 \}\ \&\ \text{PTS} \geq n\ ]\ ,$$
$$\equiv \neg\text{PTS} \geq 19$$
$$\neg\exists n[\ n \in \mathbb{N}-\{19\}\ \&\ \text{PTS} \geq n\ ]\ ,$$
$$\equiv \neg\text{PTS} \geq 1$$
$$\ldots\ \}$$
$$= \{\ \underline{\neg\text{PTS} \geq 21}\ ,\ \neg\text{PTS} \geq 20\ ,\ \neg\text{PTS} \geq 19\ ,\ \ldots\ ,\ \neg\text{PTS} \geq 1\ \}$$
$$= \{\ \neg\exists n[\ n \in Q\ \&\ \text{PTS} \geq n\ ] \mid Q \supseteq \{ m \mid m \geq 21 \}\ \}$$

$$\text{BEL}(\text{PTS} \geq 20), \text{BEL}(\text{PTS} \geq 19), \ldots \quad \text{(Strong Quantity)}$$

In combination with the Quality implicature $\text{BEL}(\neg\text{PTS} \geq 21)$, the resulting strong Quantity implicatures erroneously predict (1a) to convey that LeBron scored exactly 20 points.[11]

[11] Nouwen (2008) has claimed that exactly this sort of lower-bounded interpretation occurs with modified numerals of the form *no more than n*. If this is correct, then one of the differences between *at most* and *no more than* will be the presence vs. the absence of the *exh* operator.

Finally, this proposal also sheds light on the pragmatic existential inference that often accompanies *at most*. The relevant observation is that there is one stronger competitor to (62) that lacks a symmetric counterpart, namely the strongest competitor, ¬PTS ≥ 1. Nothing prevents the derivation of a strong Quantity implicature regarding this competitor, as it does not contradict the results of any other weak Quantity implicature—for instance, it is consistent with POSS(PTS = 20), POSS(1 ≤ PTS ≤ 19), etc.:

(i)	BEL(¬PTS ≥ 21)	(Quality)
(ii)	¬BEL(¬PTS ≥ 1)	(Weak Quantity)
(iii)	BEL(¬PTS ≥ 1) ∨ BEL(PTS ≥ 1)	(Competence)
(iv)	BEL(PTS ≥ 1)	(Strong Quantity)

Of course, this strong Quantity implicature amounts to the desired existential inference. Such an account accords with the preceding observation that this inference patterns with other strong Quantity implicatures, such as the exlusivity implicature of ordinary disjunction.

4. Ignorance, variation, and scope

Our discussion so far has focused on how *at least* and *at most* manage to express speaker ignorance. In this section, I take up the variation readings that emerge in the presence of modals and universal quantifiers. I begin in section 4.1 by considering these operators' interactions with universal modals, which are relatively well-understood. In sections 4.2 and 4.3, I turn to two interactions which have so far resisted successful analysis: (i) the ignorance inferences that arise with universal DPs, and (ii) the variation inferences observed for *at most* with existential modals. There, I show how a novel aspect of the present proposal, namely the embedded occurrence of *exh*, allows for a satisfactory account of these readings.

4.1 Universal operators and the obviation of symmetry

It is well-known that when either ordinary disjunction or *at least* occurs with a universal modal, two interpretations are possible—an ignorance reading regarding what is minimally necessary or required, and more saliently, a variation reading regarding what is sufficient or permissible:

(64) (To pass this class,) you <u>need</u> to take a final exam or write a term paper…
a. but I can't remember what I wrote on the syllabus. (ignorance)
b. and it's enough to complete just one of them. (variation)

(65) (In order to win the scoring title,) <u>LeBron</u> needs to score at least 45 points tonight…
a. but I don't know exactly how many he needs. (ignorance)
b. he already has 38 points, so he only needs 7 more! (variation)

Another pleasing feature of the Standard Recipe is the way that it derives these two interpretations as a simple matter of scope (Fox 2007, Büring 2008). When ordinary disjunction scopes over a universal modal, the individual disjuncts transparently form a pair of symmetric stronger competitors, producing the ignorance reading in (64a):

(66) [You need to take a final exam] or [you need to write a term paper].

$\text{COMP}(66) = \{ \ \underline{\Box E \vee \Box P}\ , \underbrace{\Box E\ ,\ \Box P}\ , \ldots \ \}$

$\text{POSS}(\Box E)\ \&\ \text{POSS}(\neg\Box E)$ (Ignorance)
$\text{POSS}(\Box P)\ \&\ \text{POSS}(\neg\Box P)$ (Ignorance)

Likewise, the ignorance reading in (65a) is produced when *at least* and *exh* both scope over the modal. The scopal order $exh \succ \Box$ in the prejacent of *at least* results in a statement concerning the minimum required value—it is necessary that LeBron score 20 points, but not that he score 21 points or 22 points or … . Since it follows that the maximum number of points that he reaches in every accessible world is 20, I represent this meaning as $\text{MAX}_n[\Box\text{PTS} \geq n] = 20$, and similarly for the prejacent's focus alternatives:

(67) $[\![$ at least$_C$ $[_S$ *exh* [need [LeBron scored [20 points]$_F$]]] $]\!]$ $(C = [\![\ S\ ]\!]_f)$

$= (\Box\text{PTS} \geq 20\ \&\ \neg\Box\text{PTS} \geq 21\ \&\ \ldots) \vee (\Box\text{PTS} \geq 21\ \&\ \neg\Box\text{PTS} \geq 22\ \&\ \ldots) \vee$

$\equiv \text{MAX}_n[\Box\text{PTS} \geq n] = 20$ $\qquad \equiv \text{MAX}_n[\Box\text{PTS} \geq n] = 21$

$(\Box\text{PTS} \geq 22\ \&\ \neg\Box\text{PTS} \geq 23\ \&\ \ldots) \vee \ldots$

$\equiv \text{MAX}_n[\Box\text{PTS} \geq n] = 22$

$\equiv \exists n[\ n \in \{\ m \mid m \geq 20\ \}\ \&\ \text{MAX}_m[\Box\text{PTS} \geq m] = n\ ]$

$\equiv \text{MAX}_n[\Box\text{PTS} \geq n] \geq 20$

The stronger pragmatic competitors to (67) form symmetric pairs, and so the final interpretation conveys ignorance regarding the minimum required value:

$\text{COMP}(67) = \{\ \underline{\exists n[\ n \in \{\ m \mid m \geq 20\ \}\ \&\ \text{MAX}_m[\Box\text{PTS} \geq m] = n\ ]}\ ,$

$\equiv \text{MAX}_n[\Box\text{PTS} \geq n] \geq 20$

$\exists n[\ n \in \{\ m \mid m \geq 20\ \} - \{\ 20\ \}\ \&\ \text{MAX}_m(\Box[\ \text{PTS} \geq m\ ]) = n\ ]\ ,$

$\equiv \text{MAX}_n[\Box\text{PTS} \geq n] \geq 21$

$\exists n[\ n \in \{\ 20\ \}\ \&\ \text{MAX}_n[\Box\text{PTS} \geq n] = n\ ]\ ,$

$\equiv \text{MAX}_n[\Box\text{PTS} \geq n] = 20$

$\ldots\ \}$

$= \{\ \exists n[\ n \in Q\ \&\ \text{MAX}_m[\Box\text{PTS} \geq m] = n\ ]\ \mid\ Q \subseteq \{\ m \mid m \geq 20\ \}\ \}$

$\text{POSS}(\text{MAX}_n[\Box\text{PTS} \geq n] = 20)$ & $\text{POSS}(\neg\text{MAX}_n[\Box\text{PTS} \geq n] = 20)$
$\text{POSS}(\text{MAX}_n[\Box\text{PTS} \geq n] = 21)$ & $\text{POSS}(\neg\text{MAX}_n[\Box\text{PTS} \geq n] = 21)$
…

$\forall n[\ n \geq 20 \rightarrow (\text{POSS}(\text{MAX}_m[\Box\text{PTS} \geq m] = n)\ \&\ \text{POSS}(\neg\text{MAX}_m[\Box\text{PTS} \geq m] = n))\ ]$
(Ignorance re: prejacent and (contextually relevant) higher alternatives)

The variation readings in (64b) and (65b) are produced when the modal scopes over disjunction and *at least* (plus *exh*):

(68) need [you take a final exam] or [you write a term paper].

COMP(68) = { $\underline{\Box[\text{E} \vee \text{P}]}$, $\Box\text{E}$, $\Box\text{P}$, … }

In (68), the scopal order $\Box \succ \vee$ has the effect of obviating the symmetry relationship between the individual disjuncts. Consequently, strengthening of the weak Quantity implicatures regarding these disjuncts may proceed without any contradiction—from one's certainty that you need to either take an exam or write a paper, and that you do not have to take the exam, it does not follow that one is certain that you have to write a paper, $\text{BEL}(\Box\text{P})$:

(i)	$\text{BEL}(\Box[\text{E} \vee \text{P}])$	(Quality)
(ii)	$\neg\text{BEL}(\Box\text{E})$, $\neg\text{BEL}(\Box\text{P})$	(Weak Quantity)
(iii)	$\text{BEL}(\Box\text{E}) \vee \text{BEL}(\neg\Box\text{E})$	(Competence)
(iv)	$\text{BEL}(\neg\Box\text{E})$	(Strong Quantity)

Rather, what follows is one's certainty that you are <u>allowed</u> to write a paper, and analogous reasoning yields that you are also allowed to take the exam:

(v)	$\text{BEL}(\Diamond\text{P}), \text{BEL}(\Diamond\text{E})$	(Entailments)

Together, these inferences convey variation in the permitted options for finishing the course. The situation is no different for the scopal order $\Box \succ$ *at least*, which obviates any symmetry amongst the stronger-domain competitors to (69). The final interpretation conveys variation in what will suffice for LeBron to win the scoring title:

(69) ⟦ need [at least$_C$ [$_S$ *exh* [LeBron scored [20 points]$_F$]]] ⟧ (C = ⟦ S ⟧$_f$)
= $\Box\text{PTS} \geq 20$

$\text{COMP}(69) = \{ \ \underline{\Box\text{PTS} \geq 20}\ , \Box\text{PTS} \geq 21\ , \Box\text{PTS} = 20\ , \ldots \ \}$

$= \{ \ \Box\exists n[\ n \in Q\ \&\ \text{PTS} = n\ ]\ \big|\ Q \subseteq \{\ m \mid\ m \geq 20\ \}\ \}$

$\text{BEL}(\Diamond\text{PTS} = 20), \text{BEL}(\Diamond\text{PTS} = 21), \ldots$

$\forall n[\ n \geq 20 \rightarrow \text{BEL}(\Diamond\text{PTS} = n)\ ]$

(Variation re: prejacent and (contextually relevant) higher alternatives)

Exactly the same scope configurations underlie the ignorance (*at most* ≻ *exh* ≻ □) and variation (□ ≻ *at most* ≻ *exh*) readings when *at most* occurs with a universal modal. Example (70a) most naturally receives an ignorance reading regarding what is minimally necessary. Example (70b), from a talk presented at the 2016 Boston University Conference on Language Development, conveys variation in what will suffice for a child to be regarded as an *n*-knower (i.e., an individual that has grasped the numeric concept *n*).

(70) a. (To win the scoring title,) LeBron needs to score at most 20 points tonight. (ignorance)
b. (from a talk at last fall's Boston Univ. Conf. on Lang. Development) To be an *n*-knower, a child had to give *n* correctly 2 out of 3 times, and give *n* in response to a different cardinal at most once. (variation)

McNabb & Penka (2014) (see also Penka 2015) have demonstrated experimentally that the variation inferences arising with *at most* are relatively harder to access than those arising with *at least*. This difference arguably follows from the above treatment of *at most* as a negated existential quantifier over propositions. Universal deontic modals in English are known to exhibit differing scopal preferences with respect to negative quantifiers (for recent discussion, see Alrenga & Kennedy 2014). The modals *need* (*to*) and *have* (*to*) generally prefer to scope below negative quantifiers, though wide scope for these modals is not completely ruled out:

(71) a. You need to do no more than ten push-ups.
b. You have to do no more than ten push-ups.
'There is no requirement that you do more than …' (preferred)
'What is required is that you do no more than …' (possible)

On the other hand, the modals *supposed* (*to*) and *should* are obliged to scope over negative quantifiers:

(72) a. You are supposed to do no more than ten push-ups.
b. You should do no more than ten push-ups.
'What is required is that you do no more than ten …' (only possibility)

It is thus predicted that modals of the former sort should preferentially give rise to ignorance readings, and indeed, the experimental items described by Penka &

McNabb involve the modals *have* (*to*) and *required* (*to*), which exhibits the same preference for narrow scope relative to negative quantifies. A further prediction is that the latter modals should only scope over *at most* (and *exh*), and so should only participate in variation readings. This expectation is borne out by the attested examples in (73), which are clearly intended to convey variation, and seem to preclude any ignorance readings.

(73) a. When I started writing for the Internet back in 1997, the general thinking amongst what few net journalists there were back then was that shorter was better. Stories were supposed to be at most a few paragraphs long.
b. (from SALT 26 call for abstracts)
The main text should be at most 3 pages in length.

Conversely, when the NPI auxiliary *need* is licensed by *at most*, only an ignorance reading is possible, a fact which follows if *need* must scope below *at most* in order to be succesfully licensed. For this reason, the substitution of NPI *need* in (74b) for *should* in (73b) is infelicitous, as it suggests that the conference organizers are not fully aware of their own abstract guidelines.

(74) a. (To win the scoring title,) LeBron need score at most 20 points tonight.
b. #The main text need be at most 3 pages in length.

Below, I illustrate how the Standard Recipe applies to the scope configuration $\Box \succ$ *at most* $\succ$ *exh*. (I have equivalently rendered the competitors to (75) as conjunctions of prohibitions, $\Box\neg(\text{A} \vee \text{B}) \equiv \Box\neg\text{A} \;\&\; \Box\neg\,\text{B}$, since this makes the pragmatic reasoning easier to follow.)

(75) $[\![$ should [at most$_C$ [$_S$ *exh* [the main text be [3 pages]$_F$ in length]]] $]\!]$ ($C = [\![\text{ S }]\!]_f$)

$$\begin{aligned}\text{COMP(75)} = \{\; & \Box\neg\text{LENGTH} \geq 4\;, \\ & \Box\neg(\text{LENGTH} \geq 4 \vee \text{LENGTH} = 3)\;, \\ & \equiv\; \Box\neg\text{LENGTH} \geq 4 \;\&\; \Box\neg\text{LENGTH} = 3 \\ & \Box\neg(\text{LENGTH} \geq 4 \vee 1 \leq \text{LENGTH} \leq 2)\;, \\ & \equiv\; \Box\neg\text{LENGTH} \geq 4 \;\&\; \Box\neg(1 \leq \text{LENGTH} \leq 2) \\ & \ldots \\ & \Box\neg\text{LENGTH} \geq 1 \;\} \\ = \{\; & \Box\neg\exists n[\, n \in Q \;\&\; \text{LENGTH} = n \,] \;\big|\; Q \supseteq \{\, m \mid m \geq 4 \,\}\,\}\end{aligned}$$

Regarding the first two stronger competitors to (75), the Standard Recipe derives the entailments in (iii)—the speaker considers it possible that 3 pages is an allowable length, and also that a length between 1 and 2 pages is allowed:

(i) $\text{BEL}(\Box\neg\text{LENGTH} \geq 4)$ (Quality)

(ii) $\neg$BEL($\Box\neg$LENGTH ≥ 4 & $\Box\neg$LENGTH $= 3$) , (Weak Quantity)
$\neg$BEL($\Box\neg$LENGTH ≥ 4 & $\Box\neg(1 \leq$ LENGTH $\leq 2)$)

(iii) $\neg$BEL($\Box\neg$LENGTH $= 3$) , (Entailments of (i), (ii))
$\equiv$ POSS($\Diamond$LENGTH $= 3$)
$\neg$BEL($\Box\neg(1 \leq$ LENGTH $\leq 2)$)
$\equiv$ POSS($\Diamond(1 \leq$ LENGTH $\leq 2)$)

Strengthening either of these entailments under competence does not yield any contradiction. For instance, from the speaker's certainty that 3 pages is an allowable length, her certainty that lengths between 1 and 2 pages are prohibited, BEL($\Box\neg(1 \leq$ LENGTH $\leq 2)$), does not follow:

(iv) BEL($\neg\Diamond$LENGTH $= 3$) $\vee$ BEL($\Diamond$LENGTH $= 3$) (Competence)
(v) BEL($\Diamond$LENGTH $= 3$) (Strong Quantity)

In other words, the two competitors do not form a symmetric pair. Analogous reasoning about the other competitors eventually yields the desired variation inference:

(vi) BEL($\Diamond$LENGTH $= 2$)
(vii) BEL($\Diamond$LENGTH $= 1$)

$\forall n[\ n \leq 3 \rightarrow$ BEL($\Diamond$LENGTH $= n$)]
(Variation re: prejacent and (contextually relevant) lower alternatives)

4.2 Two types of ignorance readings and intermediate scope

An interesting feature of the present proposal is that it makes available a third scope configuration, in which a quantificational operator scopes between *at least*/*at most* and *exh*. In contrast, almost all other accounts only allow for the two configurations *at least*/*at most* $\succ$ *Op* and *Op* $\succ$ *at least*/*at most*. I would now like to show how the possibility of intermediate scope sheds new light on certain outstanding problems concerning the distribution of ignorance and variation readings.

The first problem concerns the interactions between *at least* and *at most* and universal DPs. It has been noted that a slightly different ignorance reading emerges from such combinations than was just observed for universal modals (Nouwen 2015, Alexandroupoulou 2015, Blok 2015):

(76) a. (Regarding a beauty pageant)
But the other girls are not exactly losers—they each receive at least $50 for competing and are eligible for another $15,000 in scholarships.
b. At most, each contestant will be asked [three]$_F$ questions.

In (76a) and (76b), the ignorance expressed with *at least* and *at most* does not concern the minimal value associated with any of the quantified individuals.

This is illustrated most clearly by (76b)—the truth-conditional meaning for this sentence cannot merely require that the least number of questions asked of any contestant is no greater than three, as this would not rule out the possibility that some contestants were asked more than three questions. Rather, the ignorance expressed here concerns a single, fixed value that does not vary across the quantified individuals—(76b) conveys that each contestant will be asked the same number of questions, and that this number is certainly no greater than three (though the speaker is unsure of its precise identity).

In fact, this reading is exactly the one produced by the Standard Recipe when the universal DP subjects in (76) take intermediate scope, *at least/at most* $\succ$ $\forall x \succ$ *exh*. The truth-conditional meaning that arises from this configuration for (76a) is shown in (77)—it requires that there be some $n \geq 50$ such that each girl receives exactly n dollars.

(77) $[\![$ at least$_C$ [each girl$_x$ [*exh* [x receive [\$50]$_F$]]] $]\!]$ $\quad (C = [\![\text{S}]\!]_f)$

$$= \forall x[\ \$_x = 50\ ] \vee \forall x[\ \$_x = 51\ ] \vee \forall x[\ \$_x = 52\ ] \vee \ldots$$
$$= \exists n[\ n \in \{\ m \mid m \geq 50\ \}\ \&\ \forall x[\ \$_x = n\ ]\ ]$$

The stronger competitors to (77) form symmetric pairs, and so the final interpretation conveys ignorance regarding this single amount.

$$\text{COMP}(77) = \{\ \underline{\exists n[\ n \in \{\ m \mid m \geq 50\ \}\ \&\ \forall x[\ \$_x = n\ ]\ ]},$$
$$\left.\begin{array}{l} \exists n[\ n \in \{\ m \mid m \geq 50\ \} - \{\ 50\ \}\ \&\ \forall x[\ \$_x = n\ ]\ ], \\ \equiv\ \exists n[\ n \in \{\ m \mid m \geq 51\ \}\ \&\ \forall x[\ \$_x = n\ ]\ ] \\ \exists n[\ n \in \{\ 50\ \}\ \&\ \forall x[\ \$_x = n\ ]\ ], \\ \equiv \forall x[\ \$_x = 50\ ] \end{array}\right]$$
$$\ldots\ \}$$
$$= \{\ \exists n[\ n \in Q\ \&\ \forall x[\ \$_x = n\ ]\ ]\ \mid\ Q \subseteq \{\ m \mid m \geq 50\ \}\ \}$$

$$\textsc{Poss}(\forall x[\ \$_x = 50\ ])\ \&\ \textsc{Poss}(\neg \forall x[\ \$_x = 50\ ])$$
$$\textsc{Poss}(\forall x[\ \$_x = 51\ ])\ \&\ \textsc{Poss}(\neg \forall x[\ \$_x = 51\ ])$$
$$\ldots$$

$$\forall n[\ n \geq 50 \rightarrow (\textsc{Poss}(\forall x[\ \$_x = n\ ])\ \&\ \textsc{Poss}(\neg \forall x[\ \$_x = n\ ]))\ ]$$

(Ignorance re: prejacent and (contextually-relevant) higher alternatives)

An unresolved question is why *exh* cannot outscope nominal quantifiers, even though, as we have just seen, this operator must be able to outscope modals. The fact that an ignorance reading concerning the minimum value is not available in (76) suggests that the configuration *at least/at most* $\succ$ *exh* $\succ \forall x$ is not possible with universal DPs. Of course, the corresponding configuration *at least / at most* $\succ$ *exh* $\succ \Box$ underlies the ignorance re: minimum requirement reading that is attested for universal modals. I have nothing to offer about why this difference should exist, except to note that parallel observations have been made concerning the scope-taking behavior of degree quantifiers (Kennedy 1997, Heim 2001), whose meanings are often assumed to incorporate maximization. The *exh* operator also effects maximization at the propositional level.

4.3 At most, *◊, and intermediate scope*

The second problem concerns the interactions between *at most* and existential modals. It is well-known that this combination admits of an ignorance reading regarding what is maximally possible or permitted, and more saliently, a variation reading regarding what is sufficient or permissible:

(78) Your paper is <u>allowed to</u> be at most 20 pages long …
 a. and it might even have to be shorter than that. (ignorance)
 b. and since yours is 20 pages in length, we can accept it as-is. (variation)

The Standard Recipe derives the ignorance reading in (78a) from the scope configuration *at most* ≻ *exh* ≻ ◊. Exhaustification of the prejacent this time results in a statement concerning the maximum permitted value—it is allowed that your paper be 20 pages long, but not that it be 21 pages long or 22 pages long or … . It follows that the maximum length that the paper reaches in any accessible world is 20 pages, $\text{MAX}_n[\Diamond\text{LENGTH} \geq n] = 20$. Amongst the focus alternatives to the prejacent will be $\text{MAX}_n[\Diamond\text{LENGTH} \geq n] = 19$, $\text{MAX}_n[\Diamond\text{LENGTH} \geq n] = 21$, etc. Pragmatic reasoning over the stronger-domain competitors evoked by *at most* then proceeds in the expected fashion.

Interestingly, the scopal order *at most* ≻ ◊ appears also to underlie the variation reading in (78b) (contra Coppock & Brockhagen (2013), Kennedy (2015), and following Penka (2015)). Note first that the upper-bounding inference conveyed by (78b) is semantic, rather than pragmatic, in nature:

(79) #Your paper is allowed to be at most 20 pages long, perhaps even longer.

This would not be the case if the modal instead scoped over *at most*—the truth conditions determined by the configuration ◊ ≻ *at most* ≻ *exh* are quite weak, and merely state that papers not exceeding 20 pages in length will be accepted. Second, a corresponding variation reading is possible when *at most* occurs with an existential DP headed by NPI *any*. Example (3b), repeated below as (80), conveys that variation in an individuals' donations to different election candidates is allowed.

(80) Individuals can give to as many federal candidates as they want, so long as they give at most $2600 to <u>any single candidate</u> in an election cycle.

Under the assumption that *at most* is responsible for the licensing of NPI *any*, it follows that the variation reading arises when *at most* scopes over the existential.[12]

The present proposal is able to accommodate these observations, as it also allows for the modal to take intermediate scope between *at most* and *exh.* Furthermore, application of the Standard Recipe to the scope configuration *at most* $\succ \Diamond \succ$ *exh* yields variation. Here is a simple illustration of this fact. Since *at most* introduces a negated existential quantifier over propositions, the relevant configuration is $\neg\exists p \succ \Diamond \succ exh$. Now,

$\neg\exists p \succ \Diamond \succ exh$
$\equiv\ \neg \succ \Diamond \succ \exists p \succ exh$ (via commutativity of $\Diamond$, $\exists$)
$\equiv\ \Box \succ \neg \succ \exists p \succ exh$ (via $\neg\Diamond \equiv \Box\neg$)
$\equiv\ \Box \succ \neg\exists p \succ exh$

In other words, the truth-conditional meanings determined by *at most* $\succ \Diamond \succ$ *exh* and $\Box \succ$ *at most* $\succ$ *exh* are equivalent. The latter configuration is already familiar from (75), where it yielded a variation reading. So too, then, will the first configuration. More generally, the variation readings observed when *at most* occurs with an existential modal or with a universal modal should be equivalent. This is why both (81a) and (81b) may serve to convey the same sort of variation in what constitutes an acceptableacceptable submission.

(81) a. One person <u>can</u> submit at most one abstract as sole author and one abstract as co-author (or two co-authored abstracts). (*at most* $\succ \Diamond \succ$ *exh*)
b. The main text <u>should</u> be at most 3 pages in length. ($\Box \succ$ *at most* $\succ$ *exh*)

5. (At least) some remaining questions

To briefly summarize: I have attempted to show in this paper that the problems facing the simple disjunctive analysis of *at least* are not insurmountable, but rather can be solved with recourse to familiar and independently justified assumptions. While a corresponding analysis of *at most* is not tenable, an analysis in terms of negated disjunction does appear to be. Importantly, the same pragmatic mechanisms that yield ignorance and variation for *at least* may also be brought to bear on *at most*, despite this difference.

Assuming that the above analysis is on the right track, many questions still remain. Here are some of them:

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[12] The ungrammaticality of (i) suggests that *any* in (80) is an instance of its negative polarity use, rather than its free choice use:

(i) *Individuals can give to as many federal candidates as they want, so long as they give at most $2600 to <u>almost any single candidate</u> in an election cycle.

(i) The expected readings when *at least* and *at most* scope under existential modals appear not to be attested (see, e.g., Blok 2015). Why not?
(ii) Does *exh* have a role to play when *at least* and *at most* operate over non-entailment scales? If not, then do we expect a different distribution of ignorance and variation readings for these cases?
(iii) What motivates the appearance of *exh* with *at least* (see Katzir & Singh 2013 on *or* and redundancy)? With *at most*?

(iv) What should we make of the superlative morphology exhibited by *at least* and *at most* (cf. *at worst* and *at best,* which also convey ignorance and variation)?
(v) How well does this proposal extend to other scalar modifiers that exhibit (roughly) the same pattern of ignorance and variation (see, e.g., Nouwen 2010, Rett 2015)?

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